Social Optima in Robust Mean Field LQG Control: From Finite to Infinite Horizon

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***Abstract*—This article studies social optimal control of mean field linear-quadratic-Gaussian models with uncertainty. Specially, the uncertainty is represented by an uncertain drift, which is common for all agents. A robust optimization approach is applied by assuming all agents treat the uncertain drift as an adversarial player. In our model, both dynamics and costs of agents are coupled by mean field terms, and both finite- and infinite-time horizon cases are considered. By examining social functional variation and exploiting person-by-person optimality principle, we construct an auxiliary control problem for the generic agent via a class of forward-backward stochastic differential equation system. By solving the auxiliary problem and constructing consistent mean field approximation, a set of decentralized control strategies is designed and shown to be asymptotically optimal.**

***Index Terms*—Forward-backward stochastic differential equation (FBSDE), linear quadratic optimal control, mean field control, model uncertainty, social functional variation.**

I. Introduction

A. Background and Motivation

MEAN field games and control have drawn increasing attention in many fields, including system control, applied mathematics, and economics [4], [6], [12]. The mean field game involves a very large population of small interacting players with the feature that while the influence of each one is negligible, the impact of the overall population is significant. By now, mean field games and control have been intensively studied in the linear-quadratic-Gaussian (LQG) framework [17], [18], [24], [29], [37], and there is a large body of works on nonlinear models [7], [20], [23]. Huang *et al.* designed ϵ-Nash equilibrium strategies for LQG mean field games with discount costs based on the proposed Nash certainty equivalence (NCE) approach [17], [18]. The NCE approach was then applied to the cases with long run average costs [24] and with Markov jump parameters [38], respectively. Lasry and Lions independently introduced the model of mean field games and studied well-posedness of limiting partial differential equations [23]. For further literature, readers are referred to [16], [38], and [40] on mean field games with major players, [7] on probabilistic analysis of mean field games, and [42] on the oblivious equilibrium in dynamic games.

Besides noncooperative games, social optima in mean field models have also drawn much attention. The social optimum control refers to that all the players cooperate to optimize the common social cost—the sum of individual costs, which is usually regarded as a type of team decision problem [13]. Huang *et al.* considered social optima in mean field LQG control, and provided an asymptotic team-optimal solution [19]. Wang and Zhang investigated a mean field social optimal problem where a Markov jump parameter appears as a common source of randomness [41]. Also, see [21] for social optima in mixed games, [2] for team-optimal control with finite population and partial information, and [31] for mean field limit of dynamic team problems.

Mathematical models can only be approximations of the real world. Actually, some parts of a model may be inexact. Thus, it is worthwhile to study the mean field control with model uncertainty [3]. The works [14], [15], [36] investigated the mean field games and control with a global uncertainty term. The "hard constraint" case (the disturbance is specified with a bound) was considered in [14] under which the substantial difficulty arises after the Lagrange multiplier is introduced. Huang *et al.* [15] and [36] adopted the "soft constraint" approach ([3], [5], [9]) by removing the bound of the disturbance while the effort is penalized in the cost function. The works by Moon *et al.* [29], [35] considered the case that each agent is paired with the local disturbance as an adversarial player, and provided an ϵ-Nash equilibrium by tackling a Hamilton–Jacobi–Isaacs equation combined with a fixed-point analysis.

B. Challenge and Contribution

This article investigates mean field LQG social optimum control with a common uncertain drift, where both dynamics and costs of agents involve mean field coupled terms. To address

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the model uncertainty, a minus quadratic penalty term of drift is incorporated into the cost functional. There exist some substantial challenges in studying the problem. First, different from [15] and [35], the socially optimal control with respect to drift uncertainty is a high-dimensional optimization problem with indefinite state weights. The corresponding convexity condition is very hard to verify. Second, by social variational derivation, the resulting limit system is governed by a controlled forward-backward stochastic differential equation (FBSDE). To design decentralized strategies, we need to solve the auxiliary optimal control problem subject to an FBSDE system. Meanwhile, the asymptotic optimality analysis is different from general mean field LQG problems since two sequential optimizations are involved in the soft control setup. Third, for the social optimum problem in the infinite horizon, we face with tackle infinite-horizon FBSDEs and relevant optimal control problems.

In this article, the social optimum control for the robust mean field LQG model is tackled by using stochastic maximum principle [44]–[46]. For the finite-horizon problem, we first obtain some low-dimensional convexity conditions and a set of FBSDEs by analyzing the variation of the centralized maximization cost to drift uncertainty. With the help of the Riccati equation, we further obtain a feedback type of the "worst-case" drift for the social optimum problem. Next, we construct an auxiliary optimal control problem based on the social variational derivation and the person-by-person optimality principle. By solving the auxiliary problem combined with consistent mean field approximations, a set of decentralized control laws is designed and further shown to be asymptotically robust social optimal by perturbation analysis. Finally, from asymptotic analysis to FBSDEs, we design decentralized strategies and show their robust optimality for the infinite-horizon social optimum problem.

The main contributions of the article are summarized as follows.

1) Social optimum control is studied for mean field models with a common uncertain drift, where coupled terms are included in both costs and dynamics of agents. By FBSDE and Riccati equation approaches, we design a set of decentralized feedback control laws.
2) By examining the social cost variation, we give low-dimensional convexity conditions and asymptotic convexity analysis for robust social optimum problems.
3) From consistency requirements in mean field approximations, a system of differential equations is derived. The existence condition of solutions to consistency equations is characterized by a Riccati equation, instead of fixed-point analysis.
4) From perturbation analysis to FBSDE, decentralized strategies are shown to have asymptotic robust optimality.
5) By analyzing asymptotic behavior of FBSDE, decentralized strategies for the infinite-horizon problem are designed and further shown be robust social optimal.

C. Organization and Notation

The organization of the article is as follows. In Section II, we consider the finite-horizon social optimization problem with drift uncertainty. By variational analysis, the centralized control with respect to drift uncertainty is obtained. Then, an auxiliary optimal control problem is constructed based on person-by-person optimality. By solving this problem combined with consistent mean field approximations, a set of decentralized strategies is designed and further proved to be robust social optimal. Section III tackles the infinite-horizon social optimum problem. In Section IV, a numerical example is provided to verify the result. Section V concludes the article.

Notation: Suppose that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$ is a complete filtered probability space. Denote by $\otimes$ the Kronecker product, I_m m-dimensional identity matrix. We use $\|\cdot\|$ to denote the norm of a Euclidean space, or the Frobenius norm for matrices. For a symmetric matrix Q and a vector z, $\|z\|_Q^2 = z^TQz$; for two vectors x, y, $\langle x, y\rangle = x^Ty$. For a matrix (vector) M, M^T denotes its transpose, $M > 0$ means that M is positive definite. Let $L^2_{\mathcal{F}}(0,T;\mathbb{R}^k)$ denote the space of all $\mathbb{R}^k$-valued $\mathcal{F}_t$-progressively measurable processes $x(\cdot)$ satisfying $\mathbb{E}\int_0^T \|x(t)\|^2 dt < \infty$, and $L^2_{\mathcal{F},\frac{\rho}{2}}(0,\infty;\mathbb{R}^k)$ denote the space of all $\mathbb{R}^k$-valued $\mathcal{F}_t$-progressively measurable processes $x(\cdot)$ satisfying $\mathbb{E}\int_0^\infty e^{-\rho t}\|x(t)\|^2 dt < \infty$. $C([0,T],\mathbb{R}^k)$ is the space of all $\mathbb{R}^k$-valued functions defined on $[0,T]$, which are continuous; $C_{\rho/2}([0,\infty),\mathbb{R}^k)$ is a subspace of $C([0,\infty),\mathbb{R}^k)$ which is given by $\{f| \int_0^\infty e^{-\rho t}\|f(t)\|^2 dt < \infty\}$. For convenience of presentation, we use C (or $C_1, C_2, \ldots$) to denote a generic constant, which may vary from place to place.

II. Robust Mean Field Social Control Over a Finite Horizon

Consider a large population systems with N agents. The ith agent evolves by the following stochastic differential equation:

$$dx_i(t) = [Ax_i(t) + Bu_i(t) + Gx^{(N)}(t) + f(t)]dt + \sigma dW_i(t),\ 1 \le i \le N \tag{1}$$

where $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^r$ are the state and the input of agent i, respectively. $x^{(N)}(t) = \frac{1}{N}\sum_{j=1}^N x_j(t)$. $\{W_i, 1 \le i \le N\}$ are a sequence of mutually independent d-dimensional Brownian motions. $f \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$ is an unknown disturbance, which reflects the effect imposed to each agent by the eternal environment. The cost function of agent i is given by

$$J_i^{\mathrm{F}}(u) = \frac{1}{2}\mathbb{E}\int_0^T \Big\{\big\|x_i(t) - \Gamma x^{(N)}(t) - \eta\big\|_Q^2 + \|u_i(t)\|_{R_1}^2 - \|f(t)\|_{R_2}^2\Big\} dt + \frac{1}{2}\mathbb{E}\|x_i(T)\|_H^2 \tag{2}$$

where $Q, R_1, R_2, H \in \mathbb{R}^{n\times n}$ are symmetric, $\Gamma \in \mathbb{R}^{n\times n}$ and $\eta \in \mathbb{R}^n$. $u = \{u_1, \ldots, u_N\}$. Take $\{\mathcal{F}_t\}_{0\le t\le T}$ as the natural filtration generated by the Nd-dimensional Brownian motion $(W_1, \ldots, W_N)$. Denote $J_{\mathrm{soc}}^{\mathrm{F}}(u) = \sum_{i=1}^N J_i^{\mathrm{F}}(u)$. Let the social cost under the worst-case disturbance be

$$J_{\mathrm{soc}}^{\mathrm{wo}}(u) = \sup_{f\in\mathcal{U}_c^{\mathrm{F}}} J_{\mathrm{soc}}^{\mathrm{F}}(u, f). \tag{3}$$

Define the centralized control set as

$$\mathcal{U}_c^{\mathrm{F}} = \Big\{u_i|\ u_i(t) \in \sigma(x_i(0), W_i(s), 0 \le s \le t,\ 1 \le i \le N)\ \ \mathbb{E}\int_0^T \|u_i(t)\|^2 dt < \infty\Big\}$$

and the decentralized control set as

$$\mathcal{U}_{d,i}^{\mathrm{F}} = \Big\{u_i \mid u_i(t) \in \sigma(x_i(s), W_i(s), 0 \le s \le t)\ \ \mathbb{E}\int_0^T \|u_i(t)\|^2 dt < \infty\Big\}.$$

Initially, we consider the following problem.

Problem (PF): Seek a set of centralized control laws $(\check{u}_1, \ldots, \check{u}_N)$ to minimize the social cost under the worst-case disturbance for System (1)–(3), i.e., $\inf_{u_i \in \mathcal{U}_c^{\rm F}} J_{\rm soc}^{\rm wo}(u)$.

Due to accessible information restriction and high computational complexity, one generally cannot attain centralized social optima, but asymptotic social optima under decentralized control, i.e., the optimality loss tends to 0, when $N \to \infty$. Thus, we mainly study the following problem in this article.

Problem (PFa): Seek a set of decentralized control laws $(\hat{u}_1, \ldots, \hat{u}_N)$ in $\mathcal{U}_{d,i}^{\rm F}$ to asymptotically optimize the social cost under the worst-case disturbance for System (1)–(3), i.e.,

$$\left| J_{\rm soc}^{\rm wo}(\hat{u}) - \inf_{u_i \in \mathcal{U}_c^{\rm F}} J_{\rm soc}^{\rm wo}(u) \right| = o(1).$$

Remark 2.1: Different from [15] and [35], we assume the disturbance f is a common stochastic process. Here, f may denote the impact from tax, subsidy, or physical factors. Thus, agents may be conservative to anticipate the disturbance would use the information of all agents to play against them.

Remark 2.2: The notations $J_i^{\rm F}$, $J_{\rm soc}^{\rm F}$, and $J_{\rm soc}^{\rm wo}$ are actually dependent on N. However, for sake of expression simplicity, here we will not explicitly write out N. This is also applicable to $\check{J}_{\rm soc}^{\rm F}$, J_i, $J_{\rm soc}, \ldots$ in the following sections.

We make two assumptions as follows.

(A0) $\{x_i(0)\}$ are independent random variables with the same mathematical expectation. $x_i(0) = x_{i0}$, $\mathbb{E}x_i(0) = \bar{x}_0, 1 \le i \le N$. There exists a constant C_0 such that $\max_{1 \le i \le N} \mathbb{E}\|x_{i0}\|^2 < C_0$. Furthermore, $\{x_{i0}, i = 1, \ldots, N\}$ and $\{W_i, i = 1, \ldots, N\}$ are independent of each other.

(A1) $Q \ge 0$, $R_1 > 0$, $R_2 > 0$, and $H \ge 0$.

From now on, the time variable t might be suppressed if necessary and no confusion occurs.

A. Control Problem With Respect to the Disturbance

Let $u_i = \check{u}_i \in \mathcal{U}_c^{\rm F}, i = 1, \ldots, N$ be fixed. The optimal control problem with respect to the disturbance is as follows:

$$\textbf{(P1)} \text{ maximize}_{f \in \mathcal{U}_c^{\rm F}} J_{\rm soc}^{\rm F}(\check{u}, f).$$

Clearly, (P1) is equivalent to the following problem:

$$(\mathbf{P1'})\mathbf{minimize}_{\mathbf{f} \in \mathcal{U}_c^{\rm F}} \check{\mathbf{J}}_{\rm soc}^{\rm F}(\mathbf{f})$$

where

$$\check{J}_{\rm soc}^{\rm F}(f) = \frac{1}{2}\sum_{i=1}^N \mathbb{E}\int_0^T \left\{ -\left\|x_i - \Gamma x^{(N)} - \eta\right\|_Q^2 + \|f\|_{R_2}^2 \right\} dt - \frac{1}{2}\mathbb{E}\|x_i(T)\|_H^2.$$

Let $\mathbf{x} = (x_1^T, \ldots, x_N^T)^T, \mathbf{u} = (u_1^T, \ldots, u_N^T)^T, \mathbf{1} = (1, \ldots, 1)^T, \mathbf{W} = (W_1^T, \ldots, W_N^T)^T, \mathbf{A} = \text{diag}(A, \ldots, A), \mathbf{B} = \text{diag}(B, \ldots, B), \hat{\sigma} = \text{diag}(\sigma, \ldots, \sigma), \mathbf{H} = \text{diag}(H, \ldots, H), \hat{\mathbf{Q}} = \text{diag}\{Q, \ldots, Q\} - \frac{1}{N}\mathbf{1}\mathbf{1}^T \otimes \Psi$, and $\hat{\eta} = \mathbf{1} \otimes \bar{\eta}$, where $\Psi \triangleq \Gamma^T Q + Q\Gamma - \Gamma^T Q \Gamma$ and $\bar{\eta} \triangleq Q\eta - \Gamma^T Q \eta$. We can write Problem (P1′) as to minimize

$$\check{J}_{\rm soc}^{\rm F}(f) = \frac{1}{2}\mathbb{E}\int_0^T \left(-\mathbf{x}^T \hat{\mathbf{Q}}\mathbf{x} + 2\hat{\eta}^T \mathbf{x} + N f^T R_2 f \right) dt - \frac{1}{2}\mathbb{E}[\mathbf{x}^T(T)\mathbf{H}\mathbf{x}(T)]$$

subject to

$$d\mathbf{x}(t) = \check{\mathbf{A}}\mathbf{x}(t)dt + \mathbf{B}\mathbf{u}(t)dt + \mathbf{1} \otimes f(t)dt + \hat{\sigma}d\mathbf{W}(t).$$

where $\check{\mathbf{A}} \triangleq \mathbf{A} + \frac{1}{N}(\mathbf{1}\mathbf{1}^T \otimes G)$.

For the further existence analysis, we introduce the following assumptions:

(A2) Problem (P1′) is convex in f;

(A2′) Problem (P1′) is uniformly convex in f.

Below are some necessary and sufficient conditions to ensure (A2) or (A2′).

Proposition 2.1: The following statements are equivalent:

i) Problem (P1′) is convex in f.

ii) For any $f \in \mathcal{U}_c^{\rm F}$

$$\int_0^T \left(-\mathbf{y}^T \hat{\mathbf{Q}}\mathbf{y} + N f^T R_2 f \right) dt - \|\mathbf{y}(T)\|_{\mathbf{H}}^2 \ge 0$$

where $\mathbf{y} \in \mathbb{R}^{nN}$ satisfies

$$d\mathbf{y} = (\check{\mathbf{A}}\mathbf{y} + \mathbf{1} \otimes f)dt, \ \mathbf{y}(0) = 0.$$

iii) For any $f \in \mathcal{U}_c^{\rm F}$

$$\int_0^T \left\{ -\left\|(I - \Gamma)y_i\right\|_Q^2 + \|f\|_{R_2}^2 \right\} dt - \|y_i(T)\|_H^2 \ge 0$$

where for $i = 1, 2, \ldots, N$, y_i satisfies

$$dy_i = [Ay_i + Gy^{(N)} + f]dt, \quad y_i(0) = 0. \tag{4}$$

Proof: (i) ⇔ (ii) is given in [15] and [25]. From (4), we have $y_1 = y_2 = \cdots = y_N = y^{(N)}$. Thus,

$$\begin{aligned} &\int_0^T \left(-\mathbf{y}^T \hat{\mathbf{Q}}\mathbf{y} + N f^T R_2 f \right) dt - \|\mathbf{y}(T)\|_{\mathbf{H}}^2 \\ &= \sum_{i=1}^N \int_0^T \left(-\left\|y_i - \Gamma y_i\right\|_Q^2 + \|f\|_{R_2}^2 \right) dt - \sum_{i=1}^N \|y_i(T)\|_H^2 \\ &= N\left[\int_0^T \left(-\left\|(I-\Gamma)y_i\right\|_Q^2 + \|f\|_{R_2}^2 \right) dt - \|y_i(T)\|_H^2 \right] \end{aligned} \tag{5}$$

which implies that (ii) is equivalent to (iii). ■

Proposition 2.2: The following statements are equivalent

i) Problem (P1′) is uniformly convex in f.

ii) There exists $\delta > 0$ such that

$$\int_0^T \left(-\mathbf{y}^T \hat{\mathbf{Q}}\mathbf{y} + N f^T R_2 f \right) dt - \|\mathbf{y}(T)\|_{\mathbf{H}}^2 \ge \delta N \int_0^T \|f\|^2 dt.$$

iii) The equation

$$\dot{\mathbf{P}} + \check{\mathbf{A}}^T \mathbf{P} + \mathbf{P}\check{\mathbf{A}} - \hat{\mathbf{Q}} - \mathbf{P}(\mathbf{1} \otimes I)(NR_2)^{-1}(\mathbf{1}^T \otimes I)\mathbf{P} = 0$$

with $\mathbf{P}(T) = -\mathbf{H}$ admits a solution in $C([0,T]; \mathbb{R}^{nN})$.

iv) The following equation admits a solution in $C([0,T]; \mathbb{R}^n)$

$$\dot{P} + (A+G)^T P + P(A+G) - PR_2^{-1}P - (I - \Gamma)^T Q(I - \Gamma) = 0, \quad P(T) = -H.$$

v) For any $t \in [0,T]$, $\det[(0,I)e^{\mathbb{A}t}(0,I)^T] > 0$, where

$$\mathbb{A} = \begin{pmatrix} A + G + R_2^{-1}H & -R_2^{-1} \\ \mathbb{A}_{21} & -(A + G + R_2^{-1}H)^T \end{pmatrix}$$

with $\mathbb{A}_{21} = HR_2^{-1}H + (I - \Gamma)^T Q(I - \Gamma) + (A+G)^T H + H(A+G)$.

Proof: (i)⇔(ii) is implied from [15] and [25]. (i)⇔(iii) is given by [33, Th. 4.5]. By (5) and (ii), we have

$$\int_0^T \left\{ -\|y_i\|_{Q(I-\Gamma)}^2 + \|f\|_{R_2}^2 \right\} dt - \|y_i(T)\|_H^2 \ge \frac{\delta}{N}\int_0^T \|f\|^2 dt.$$

By [33, Th. 4.5], we obtain (ii)⇔(iv), which further implies (i)⇔(iv). (iv)⇔(v) is given by [15] and [27]. ■

By variational analysis, we obtain necessary and sufficient conditions for the existence of centralized minimizer of (P1′).

Theorem 2.1: Assume (A0)–(A1) hold. (P1′) has a minimizer in $\mathcal{U}_c^{\rm F}$ if and only if (A2) holds and the following equations admit a set of solutions $(\check{x}_i, \check{p}_i, \check{\beta}_i^j, i, j = 1, \ldots, N)$:

$$\begin{cases} d\check{x}_i = (A\check{x}_i + B\check{u}_i + G\check{x}^{(N)} - R_2^{-1}\check{p}_j^{(N)})dt + \sigma dW_i \\ d\check{p}_i = -[A^T\check{p}_i + G^T\check{p}_j^{(N)} - Q\check{x}_i + \Psi\check{x}^{(N)} + \bar{\eta}]dt \\ \qquad + \sum_{j=1}^N \check{\beta}_i^j dW_j \\ \check{x}_i(0) = x_{i0},\ \check{p}_i(T) = -H\check{x}_i(T),\ 1 \le i \le N \end{cases} \tag{6}$$

where $\check{p}^{(N)} = \frac{1}{N}\sum_{j=1}^N \check{p}_j$, and furthermore the minimizer is $\check{f} = -R_2^{-1}\check{p}^{(N)}$.

Proof: Suppose that $\check{f}$ is a candidate of the minimizer of (P1′). Denote by $\check{x}_i$ the state of agent i under the control $\check{u}_i$ and the drift $\check{f}$. For any $f \in \mathcal{U}_c^{\rm F}$ and $\varepsilon \in \mathbb{R}$, let $f^\varepsilon = \check{f} + \varepsilon f$. Let x_i^ε be the solution of the following perturbed state equation:

$$dx_i^\varepsilon = \left(Ax_i^\varepsilon + B\check{u}_i + \check{f} + \varepsilon f + \frac{G}{N}\sum_{i=1}^N x_i^\varepsilon\right)dt + \sigma dW_i$$

with $x_i^\varepsilon(0) = x_{i0}$, $i = 1, 2, \ldots, N$. Let $y_i = (x_i^\varepsilon - \check{x}_i)/\varepsilon$, and $y^{(N)} = \sum_{i=1}^N y_i/N$. It can be verified that y_i satisfies (4). Let $\{\check{p}_i, \check{\beta}_i^j, i, j = 1, \ldots, N\}$ be a set of solutions to BSDE in (6). Then, by Itô's formula,

$$\begin{aligned} &-\mathbb{E}\langle H\check{x}_i(T), y_i(T)\rangle \\ &= \mathbb{E}\int_0^T \Big[\langle -(A^T\check{p}_i + G^T\check{p}_j^{(N)} - Q\check{x}_i + \Psi\check{x}^{(N)} + \bar{\eta}), y_i\rangle \\ &\quad + \langle p_i, Ay_i + Gy^{(N)}f\rangle\Big]dt. \end{aligned} \tag{7}$$

We have

$$\check{J}_{\rm soc}^{\rm F}(\check{f} + \varepsilon f) - \check{J}_{\rm soc}^{\rm F}(\check{f}) = \varepsilon\Lambda_1 + \frac{\varepsilon^2}{2}\Lambda_2 \tag{8}$$

where

$$\begin{aligned} \Lambda_1 \triangleq &\sum_{i=1}^N \mathbb{E}\int_0^T \Big[\big\langle -Q(\check{x}_i - (\Gamma\check{x}^{(N)} + \eta)), y_i - \Gamma y^{(N)}\big\rangle \\ &+ \langle R_2\check{f}, f\rangle\Big]dt - \sum_{i=1}^N \mathbb{E}\langle H\check{x}_i(T), y_i(T)\rangle \end{aligned}$$

$$\Lambda_2 \triangleq \sum_{i=1}^N \mathbb{E}\int_0^T \Big\{-\big\|y_i - \Gamma y^{(N)}\big\|_Q^2 + \|f\|_{R_2}^2 dt - \|y_i(T)\|_H^2\Big\}.$$

Note that

$$\begin{aligned} &\sum_{i=1}^N \mathbb{E}\int_0^T \big\langle -Q(\check{x}_i - (\Gamma\check{x}^{(N)} + \eta)), \Gamma y^{(N)}\big\rangle dt \\ &= \mathbb{E}\int_0^T \left\langle -\Gamma^T Q\sum_{i=1}^N \left(\check{x}_i - (\Gamma\check{x}^{(N)} + \eta)\right), \frac{1}{N}\sum_{j=1}^N y_j\right\rangle dt \\ &= \sum_{j=1}^N \mathbb{E}\int_0^T \big\langle -\Gamma^T Q\left((I - \Gamma)\check{x}^{(N)} - \eta\right), y_j\big\rangle dt. \end{aligned}$$

From (7), one can obtain that

$$\begin{aligned} \Lambda_1 &= \mathbb{E}\sum_{i=1}^N \int_0^T \Big[\big\langle -Q(\check{x}_i - (\Gamma\check{x}^{(N)} + \eta)) \\ &\qquad y_i - \Gamma y^{(N)}\big\rangle + \langle R_2\check{f}, f\rangle\Big]dt \\ &\quad + \sum_{i=1}^N \mathbb{E}\int_0^T \Big[\langle -(A^T\check{p}_i + G^T\check{p}_j^{(N)} - Q\check{x}_i + \Psi\check{x}^{(N)} + \bar{\eta}), \\ &\qquad y_i\rangle + \langle p_i, Ay_i + Gy^{(N)}f\rangle\Big]dt. \\ &= \mathbb{E}\int_0^T \left\langle NR_2\check{f} + \sum_{i=1}^N p_i, f\right\rangle dt \end{aligned}$$

From (8), $\check{f}$ is a minimizer to Problem (P1′) if and only if $\Lambda_2 \ge 0$ and $\Lambda_1 = 0$. By Proposition 2.1, $\Lambda_2 \ge 0$ if and only if (A2) holds. Indeed, if (A2) does not hold, the minimization problem is ill-posed (see, e.g., [33]). $\Lambda_1 = 0$ is equivalent to

$$\check{f} = -R_2^{-1}\check{p}^{(N)}.$$

Thus, we have the optimality system (6). Namely, $\Lambda_1 = 0$ if and only if (6) admits a solution $(\check{x}_i, \check{p}_i, \check{\beta}_i^j, i, j = 1, \ldots, N)$. ■

Let $\check{u}^{(N)} = \frac{1}{N}\sum_{i=1}^N \check{u}_i$, and $\check{p}^{(N)} = \frac{1}{N}\sum_{i=1}^N \check{p}_i$. It follows from (6) that

$$\begin{cases} d\check{x}^{(N)} = \big((A+G)\check{x}^{(N)} + B\check{u}^{(N)} - R_2^{-1}\check{p}^{(N)}\big)dt \\ \quad + \frac{1}{N}\sum_{i=1}^N \sigma dW_i \\ d\check{p}^{(N)} = -\big[(A+G)^T\check{p}^{(N)} + (\Psi - Q)\check{x}^{(N)} + \bar{\eta}\big]dt \\ \quad + \frac{1}{N}\sum_{i=1}^N\sum_{j=1}^N \check{\beta}_i^j dW_j \\ \check{x}^{(N)}(0) = \frac{1}{N}\sum_{i=1}^N x_{i0},\ \check{p}^{(N)}(T) = -H\check{x}^{(N)}(T). \end{cases} \tag{9}$$

Proposition 2.3: The FBSDE (6) admits a set of adapted solutions $(\check{x}_i, \check{p}_i, i = 1, \ldots, N)$ if and only if (9) admits an adapted solution $(\check{x}^{(N)}, \check{p}^{(N)})$.

Proof: If (9) admits an adapted solution $(\check{x}^{(N)}, \check{p}^{(N)})$, then (6) is decoupled. The existence of a set of solutions to (6) follows. The part of necessity is straightforward. ■

We further discuss the optimal feedback control of (P1′). Let $\check{p}^{(N)}(t) = P(t)\check{x}^{(N)}(t) + \check{s}(t), t \ge 0$, where $P \in \mathbb{R}^{n\times n}$ and $\check{s} \in \mathbb{R}^n$. Then, by (9), we have

$$\begin{aligned} d\check{p}^{(N)} &= P\Bigg[(A+G)\check{x}^{(N)} + B\check{u}^{(N)} - R_2^{-1}\check{p}^{(N)})dt \\ &\quad + \frac{1}{N}\sum_{i=1}^N \sigma dW_i\Bigg] + \dot{P}\check{x}^{(N)}dt + d\check{s} \\ &= -[(A+G)^T(P\check{x}^{(N)} + \check{s}) + (\Psi - Q)\check{x}^{(N)} \\ &\quad + \bar{\eta}]dt + \frac{1}{N}\sum_{i=1}^N\sum_{j=1}^N \check{\beta}_i^j dW_j. \end{aligned}$$

This implies

$$\begin{aligned} &\dot{P} + (A+G)^T P + P(A+G) - PR_2^{-1}P \\ &- (I - \Gamma)^T Q(I - \Gamma) = 0,\ P(T) = -H \\ &d\check{s} + [(A + \bar{G})^T\check{s} + PB\check{u}^{(N)} + \bar{\eta}]dt \end{aligned} \tag{10}$$

$$+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\frac{\sigma}{N}-\breve{\beta}_i^j\right)dW_j=0,\ \breve{s}(T)=0 \tag{11}$$

where $\bar{G}\triangleq G-R_2^{-1}P$.

By the local Lipschitz continuous property of the quadratic function, (10) must admit a unique local solution in a small time duration $[T_0,T]$. The global existence of the solution can be referred to [1] and [11]. From Proposition 2.2, we obtain that under (A2′), (10) has a unique solution in $C([0,T],\mathbb{R}^{n\times n})$.

Theorem 2.2: Under (A0), (A1), and (A2′), Problem (P1′) has a minimizer

$$\breve{f}(t)=-R_2^{-1}[P(t)\breve{x}^{(N)}(t)+\breve{s}(t)],\ t\geq 0 \tag{12}$$

where P and $\breve{s}$ are solutions of (10) and (11), respectively.

Proof: Under (A2′), (10) admits a unique solution P, which implies (11) has a unique solution $\breve{s}$ in $C([0,T],\mathbb{R}^n)$. By [27, Theorem 2.4.1], (9) admits a unique solution $(\breve{x}^{(N)},\breve{p}^{(N)})$, where $\breve{p}^{(N)}=P\breve{x}^{(N)}+\breve{s}$. From Proposition 2.3, (6) is solvable. This with Theorem 2.1 completes the proof. ■

Remark 2.3: From the above analysis, (A2′) is sufficient for solvability of (6). Indeed, from [27], (A2′) is also a necessary condition to ensure solvability of (6) holds for any $\breve{u}_i\in\mathcal{U}_c^{\mathrm{F}}$.

B. Distributed Strategy Design

After the "worst-case" drift $\breve{f}$ is applied, we have the following optimal control problem.

(P2): Minimize $J_{\mathrm{soc}}^{\mathrm{F}}(u,\breve{f}(u))$ over $\{u_i\in\mathcal{U}_c^{\mathrm{F}},i=1,\ldots,N\}$, where

$$dx_i=[Ax_i+Bu_i+Gx^{(N)}-R_2^{-1}(Px^{(N)}+s)]dt$$
$$+\sigma dW_i,\ 1\leq i\leq N \tag{13}$$

$$ds=-\left[(A+\bar{G})^Ts+PBu^{(N)}+\bar{\eta}\right]dt$$
$$+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\beta_i^j-\frac{\sigma}{N}\right)dW_j,s(T)=0 \tag{14}$$

$$J_{\mathrm{soc}}^{\mathrm{F}}(u)=\frac{1}{2}\sum_{i=1}^{N}\mathbb{E}\int_0^T\Big\{\|x_i-\Gamma x^{(N)}-\eta\|_Q^2+\|u_i\|_{R_1}^2$$
$$-\|Px^{(N)}+s\|_{R_2^{-1}}^2\Big\}dt+\frac{1}{2}\mathbb{E}\|x_i(T)\|_H^2. \tag{15}$$

We first show that Problem (P2) has the property of uniformly convexity under certain conditions.

Lemma 2.1: Assume that (A0), (A1), and (A2′) hold. Then, there exists a constant $C_0>0$ with $R_1>C_0I$ and $R_2>C_0I$ such that (P2) is uniformly convex in u, where $u_i\in\mathcal{U}_c^{\mathrm{F}}$.

Proof: Denote $\bar{\mathbf{Q}}=\mathrm{diag}\{Q,\cdots,Q\}-\frac{1}{N}\mathbf{1}\mathbf{1}^T\otimes(\Psi+PR_2^{-1}P)$, $\mathbf{R}_1=\mathrm{diag}\{R_1,\ldots,R_1\}$, $\bar{\mathbf{A}}=\mathrm{diag}\{A,\ldots,A\}+\frac{1}{N}\mathbf{1}\mathbf{1}^T\otimes(G-R_2^{-1}P)$. By a similar argument with [25], we obtain that Problem (P2) is uniformly convex if for any $u_i\in\mathcal{U}_c^{\mathrm{F}}$

$$\mathbb{E}\int_0^T\left(\mathbf{z}^T\bar{\mathbf{Q}}\mathbf{z}+\mathbf{u}^T\mathbf{R}_1\mathbf{u}-N\grave{s}^TR_2^{-1}\grave{s}\right)dt$$
$$+\mathbb{E}\|\mathbf{z}(T)\|_{\mathbf{H}}^2\geq\delta\mathbb{E}\int_0^T\|\mathbf{u}\|^2dt$$

where $\mathbf{z}\in\mathbb{R}^{nN}$ and $\grave{s}\in\mathbb{R}^n$ satisfy

$$d\mathbf{z}=(\bar{\mathbf{A}}\mathbf{z}+\mathbf{B}\mathbf{u}-\mathbf{1}\otimes R_2^{-1}\grave{s})dt,\ \mathbf{z}(0)=0 \tag{16}$$

$$d\grave{s}=-\left[(A+\bar{G})^T\grave{s}+\frac{1}{N}PB(\mathbf{1}^T\otimes I)\mathbf{u}\right]dt$$
$$+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\grave{\beta}_i^jdW_j,\ \grave{s}(T)=0. \tag{17}$$

By [44, Ch. 7] and (17)

$$\mathbb{E}\int_0^T\|\grave{s}(t)\|^2dt\leq\frac{C_1}{N^2}\mathbb{E}\int_0^T\|\mathbf{1}^T\otimes I\|^2\|\mathbf{u}(t)\|^2dt$$
$$\leq\frac{C_1}{N}\mathbb{E}\int_0^T\|\mathbf{u}(t)\|^2dt. \tag{18}$$

This with (16) leads to $\mathbb{E}\int_0^T\|\mathbf{z}\|^2dt\leq C_2\mathbb{E}\int_0^T\|\mathbf{u}\|^2dt$. Note that

$$\lambda_{\min}(\bar{\mathbf{Q}})\geq\lambda_{\min}(Q)-[\lambda_{\max}(\Psi)+\lambda_{\max}(PR_2^{-1}P)]$$
$$\geq-[\lambda_{\max}(\Psi)+\lambda_{\max}(PR_2^{-1}P)]$$

where $\lambda_{\min}(Q)$ and $\lambda_{\max}(Q)$ are smallest and largest eigenvalues of Q, respectively. From this with (18), there exists $\delta>0$ and $C_0>0$ such that for $R_1>C_0I$ and $R_2>C_0I$

$$\mathbb{E}\int_0^T\left(\mathbf{z}^T\bar{\mathbf{Q}}\mathbf{z}+\mathbf{u}^T\mathbf{R}_1\mathbf{u}-N\grave{s}^TR_2^{-1}\grave{s}\right)dt+\mathbb{E}\|\mathbf{z}(T)\|_{\mathbf{H}}^2$$
$$\geq\mathbb{E}\int_0^T\left[\mathbf{u}^T(\mathbf{R}_1-C_0I_{nN})\mathbf{u}\right]dt\geq\delta\mathbb{E}\int_0^T(\mathbf{u}^T\mathbf{u})dt.$$

■

1) Social Variational Derivation: Note that the social optimum implies the person-by-person optimality [13]. We now provide a transformation of the original social optimum problem by variational derivation and person-by-person optimality. Suppose that $\breve{u}=(\breve{u}_1,\ldots,\breve{u}_N)$ is a minimizer to Problem (P2), where $\breve{u}_j\in\mathcal{U}_c^{\mathrm{F}}$. Let $\breve{x}_j$ correspond to $\breve{u}_j$, $j=1,\ldots,N$ and $\breve{x}^{(N)}=\frac{1}{N}\sum_{j=1}^{N}\breve{x}_j$. Let $\breve{s}$ correspond to $\breve{u}_1$, …, $\breve{u}_N$. Fix $\breve{u}_{-i}=(\breve{u}_1,\ldots,\breve{u}_{i-1},\breve{u}_{i+1},\ldots,\breve{u}_N)$, and perturb u_i. Denote $\delta u_i=u_i-\breve{u}_i,\delta x_j=x_j-\breve{x}_j,\delta x^{(N)}=\frac{1}{N}\sum_{j=1}^{N}\delta x_j$ and $\delta s=s-\breve{s}$. Let the strategy variation δu_i be adapted to $\mathcal{F}_t$ and satisfy $\mathbb{E}\int_0^T\|\delta u_i\|^2dt<\infty$. Let δJ_i be the variation of J_i with respect to δu_i. By (13) and (14)

$$\frac{d\delta x_j}{dt}=A\delta x_j+\frac{\bar{G}}{N}\delta x_i+\frac{\bar{G}}{N}\sum_{k\neq i}\delta x_k-R_2^{-1}\delta s$$
$$j\neq i,\ \delta x_j(0)=0$$

$$d\delta s=-\left[(A+\bar{G})^T\delta s+\frac{1}{N}PB\delta u_i\right]dt$$
$$+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\delta\beta_i^jdW_j,\ \delta s(T)=0 \tag{19}$$

where $\bar{G}\triangleq G-R_2^{-1}P$. This implies $\delta x_j=\delta x_k$, for any $j,k\neq i$. Thus

$$\frac{d\delta x_j}{dt}=\left(A+\frac{N-1}{N}\bar{G}\right)\delta x_j+\frac{\bar{G}}{N}\delta x_i-R_2^{-1}\delta s,\delta x_j(0)=0$$

which gives that

$$\delta x_j(t)=\int_0^te^{(A+\frac{N-1}{N}\bar{G})(t-\tau)}\left(\frac{\bar{G}}{N}\delta x_i(\tau)-R_2^{-1}\delta s(\tau)\right)d\tau.$$

We further have

$$\delta x^{(N)}(t) = \frac{1}{N}\delta x_i(t) + \frac{N-1}{N}\int_0^t e^{(A+\frac{N-1}{N}\bar{G})(t-\tau)} \times \left(\frac{\bar{G}}{N}\delta x_i(\tau) - R_2^{-1}\delta s(\tau)\right) d\tau.$$

By this with (14), one can obtain

$$\begin{aligned}\delta J_i^{\rm F}(u,\check{f}) = \mathbb{E}\int_0^T &\{[\check{x}_i - \Gamma\check{x}^{(N)} - \eta]^T Q[\delta x_i - \Gamma\delta x^{(N)}]\\ &- (P\check{x}^{(N)} + s)^T R_2^{-1}(P\delta x^{(N)} + \delta s)\\ &+ \check{u}_i^T R_1\delta u_i\}dt + \mathbb{E}[x_i^T(T)H\delta x_i(T)]\end{aligned}$$

and for $j \neq i$

$$\begin{aligned}\delta J_j^{\rm F}(u,\check{f}) = \mathbb{E}\int_0^T &\Big[(\check{x}_j - \Gamma\check{x}^{(N)} - \eta)^T Q(\delta x_j - \Gamma\delta x^{(N)})\\ &-(P\check{x}^{(N)} + s)^T R_2^{-1}(P\delta x^{(N)} + \delta s)\Big] dt\\ &+ \mathbb{E}[x_j^T(T)H\delta x_j(T)].\end{aligned}$$

The above equation further implies that

$$\begin{aligned}&\sum_{j\neq i}\delta J_j^{\rm F}(u,\check{f})\\ &= \mathbb{E}\int_0^T\left(\check{x}_{-i}^{(N)} - \frac{N-1}{N}(\Gamma\check{x}^{(N)} + \eta)\right)^T Q\left[\left(I - \frac{N-1}{N}\Gamma\right)\right.\\ &\left.\cdot\int_0^t e^{(A+\frac{N-1}{N}\bar{G})(t-\tau)}(\bar{G}\delta x_i - NR_2^{-1}\delta s)d\tau - \Gamma\delta x_i\right]\\ &- (P\check{x}^{(N)} + s)^T R_2^{-1}\left[P\left(\frac{(N-1)^2}{N^2}\int_0^t e^{(A+\frac{N-1}{N}\bar{G})(t-s)}\right.\right.\\ &\left.\left.\cdot\bar{G}\delta x_i ds + \frac{N-1}{N}\delta x_i\right) + (N-1)\delta s\right] dt + \mathbb{E}\left[x_{-i}^{(N)}(T)^T\right.\\ &\left.\cdot H\int_0^T e^{(A+\frac{N-1}{N}\bar{G})(T-t)}(\bar{G}\delta x_i - NR_2^{-1}\delta s)dt\right]\end{aligned}$$

where $\check{x}_{-i}^{(N)} = \frac{1}{N}\sum_{j\neq i}\check{x}_j$. Let $\delta\psi_i = N\delta s$. Then, from (19)

$$\begin{aligned}d\delta\psi_i = &-\left[(A+\bar{G})^T\delta\psi_i + PB\delta u_i\right] dt\\ &+ \sum_{i=1}^N\sum_{j=1}^N \delta\beta_i^j dW_j, \delta\psi_i(T) = 0.\end{aligned} \tag{20}$$

Since all agents in (1) and (2) are in the symmetric setup (statistically exchangeable when no controls applied), then for large N, it is plausible to approximate $\check{x}^{(N)}$ by a deterministic function $\bar{x}$ (see [10], [22], [31]). The zero first-order variational condition combined with the mean field approximation gives

$$\begin{aligned}&\mathbb{E}\int_0^T \{(\check{x}_i - \Gamma\bar{x} - \eta)^T Q\delta x_i - [((I-\Gamma)\bar{x} - \eta)^T Q\Gamma\\ &+ (P\bar{x} + \bar{s})^T R_2^{-1}P]\,\delta x_i + [((I-\Gamma)\bar{x} - \eta)^T Q(I-\Gamma)\\ &- (P\bar{x}+\bar{s})^T R_2^{-1}P]\int_0^t e^{(A+\bar{G})(t-\tau)}(\bar{G}\delta x_i - R_2^{-1}\delta\psi_i)d\tau\\ &- (P\bar{x} + \bar{s})^T R_2^{-1}\delta\psi_i + \check{u}_i^T R_1\delta u_i + \bar{x}^T(T)He^{(A+\bar{G})(T-t)}\\ &\times(\bar{G}\delta x_i - R_2^{-1}\delta\psi_i)\}\, dt + \mathbb{E}[\check{x}_i^T(T)H\delta x_i(T)] = 0\end{aligned} \tag{21}$$

where $\bar{x} \in C([0,T],\mathbb{R}^n)$ is an approximation of $\check{x}^{(N)}$. From observation, (21) is the zero variation condition for the optimal control problem with the cost function

$$\begin{aligned}&J_i'(u_i)\\ &= \frac{1}{2}\mathbb{E}\int_0^T \Big\{x_i^T Qx_i + 2[-((I-\Gamma)\bar{x} - \eta)^T Q\Gamma\\ &\quad - (\Gamma\bar{x} + \eta)^T Q - (P\bar{x} + \bar{s})^T R_2^{-1}P]x_i\\ &\quad + 2\left[((I-\Gamma)\bar{x} - \eta)^T Q(I-\Gamma) - (P\bar{x} + \bar{s})^T R_2^{-1}P\right]\\ &\quad \times\int_0^t e^{(A+\bar{G})(t-\tau)}(\bar{G}x_i - R_2^{-1}\psi_i)d\tau\\ &\quad + \bar{x}^T(T)He^{(A+\bar{G})(T-t)}(\bar{G}x_i - R_2^{-1}\psi_i) + u_i^T R_1 u_i\\ &\quad -2(P\bar{x} + \bar{s})^T R_2^{-1}\psi_i\Big\}\, dt + \frac{1}{2}\mathbb{E}[x_i^T(T)Hx_i(T)]\\ x &= \frac{1}{2}\mathbb{E}\int_0^T\Big\{x_i^T Qx_i - 2\left[\Psi\bar{x} + \bar{\eta} + (P\bar{x} + \bar{s})^T R_2^{-1}P\right] x_i\\ &\quad + 2v^T(\bar{G}x_i - R_2^{-1}\psi_i) - 2(P\bar{x} + \bar{s})^T R_2^{-1}\psi_i\\ &\quad + u_i^T R_1 u_i\Big\}\, dt + \frac{1}{2}\mathbb{E}[x_i^T(T)Hx_i(T)]\end{aligned} \tag{22}$$

where the second equality holds by an exchange of order of the integration, and

$$\begin{aligned}v(t) \triangleq \int_t^T &e^{(A+\bar{G})^T(\tau-t)}\left[(I-\Gamma)^T Q((I-\Gamma)\bar{x} - \eta)\right.\\ &\left.-PR_2^{-1}(P\bar{x} + \bar{s})\right] d\tau + e^{(A+\bar{G})^T(T-t)}H\bar{x}(T).\end{aligned}$$

2) Mean Field Approximation: Based on (13), (20), and (22), we construct the following auxiliary optimal control problem.

(P3): minimize $\bar{J}_i^{\rm F}(u_i)$ over $u_i \in \mathcal{U}_{d,i}^{\rm F}$, where

$$\begin{aligned}d\grave{x}_i &= [A\grave{x}_i + Bu_i + G\bar{x} - R_2^{-1}(P\bar{x} + \bar{s})]dt\\ &\quad + \sigma dW_i,\ \grave{x}_i(0) = x_{i0}\end{aligned} \tag{23}$$

$$\begin{aligned}d\psi_i = &-\left[(A+\bar{G})^T\psi_i + PBu_i\right] dt\\ &+ z_i^i dW_i,\ \psi_i(T) = 0.\end{aligned} \tag{24}$$

$$\begin{aligned}\bar{J}_i^{\rm F}(u_i) = \frac{1}{2}\mathbb{E}\int_0^T &\left[\grave{x}_i^T Q\grave{x}_i - 2(P\bar{x} + \bar{s})^T R_2^{-1}P\grave{x}_i\right.\\ &- 2(\Psi\bar{x} + \bar{\eta})^T\grave{x}_i + 2v^T(\bar{G}\grave{x}_i - R_2^{-1}\psi_i)\\ &\left.-2(P\bar{x} + \bar{s})^T R_2^{-1}\psi_i + u_i^T R_1 u_i\right] dt\\ &+ \frac{1}{2}\mathbb{E}[\grave{x}_i^T(T)H\grave{x}_i(T)].\end{aligned} \tag{25}$$

Here $\bar{s}, v$ are determined by

$$\dot{\bar{s}} = -(A + G - R_2^{-1}P)^T\bar{s} - PB\bar{u} - \bar{\eta},\ \bar{s}(T) = 0 \tag{26}$$

$$\begin{aligned}\dot{v} = &-(A+\bar{G})^T v - \left[(I-\Gamma)^T Q((I-\Gamma)\bar{x} - \eta)\right.\\ &\left.-PR_2^{-1}(P\bar{x} + \bar{s})\right],\ v(T) = H\bar{x}(T)\end{aligned} \tag{27}$$

and $\bar{x}, \bar{u} \in C([0,T],\mathbb{R}^n)$ are to be determined by consistency equations later.

Theorem 2.3: Assume that (A0), (A1), and (A2′) hold. Problem (P3) has a unique optimal control

$$\hat{u}_i(t) = -R_1^{-1}B^T[k_i(t) - P(t)l(t)],\ 1 \leq i \leq N \tag{28}$$

where (l, k_i, ζ_i) is a unique adaptive solution to the following (decoupled) FBSDE:

$$dl = [(A+\bar{G})l + R_2^{-1}v + R_2^{-1}(P\bar{x} + \bar{s})]dt, l(0) = 0 \tag{29}$$

$$dk_i = -\left\{A^T k_i + Q\grave{x}_i - (\Psi\bar{x} + \bar{\eta}) - PR_2^{-1}(P\bar{x} + \bar{s}) + \bar{G}^T v\right\} dt + \zeta_i dW_i,\ k_i(T) = H\grave{x}_i(T). \quad (30)$$

Proof: Since $Q \geq 0$ and $R_1 > 0$, then from [15] and [25], (P3) is uniformly convex in u_i and there exists a unique optimal control for (P3), denoted as $\hat{u}_i$. Then

$$\begin{aligned}0 &= \delta \bar{J}_i^{\mathrm{F}}(\hat{u}_i)\\ &= \mathbb{E}\int_0^T \big[(Q\grave{x}_i - \Psi\bar{x} - \bar{\eta})^T \delta\grave{x}_i - (P\bar{x} + \bar{s})^T R_2^{-1} P\delta\grave{x}_i\\ &\quad + v^T(\bar{G}\delta\grave{x}_i - R_2^{-1}\delta\psi_i) - (P\bar{x} + \bar{s})R_2^{-1}\delta\psi_i\\ &\quad + u_i^T R_1 \delta u_i\big]\, dt + \mathbb{E}[\grave{x}_i^T(T) H \delta\grave{x}_i(T)]\end{aligned} \quad (31)$$

where $\delta u_i = u_i - \hat{u}_i$, $\delta\grave{x}_i = \grave{x}_i - \hat{x}_i$, and $\delta\psi_i = \psi_i - \hat{\psi}_i$. Note that (29) and (30) are decoupled. Given $\bar{x}, \bar{u} \in C([0,T], \mathbb{R}^n)$, (29) is a standard linear BSDE and so has a unique solution (k_i, ζ_i). Note that

$$\begin{aligned}d(\delta\grave{x}_i) &= (A\delta\grave{x}_i + B\delta u_i)dt\\ d(\delta\psi_i) &= -\left[(A + \bar{G})^T \delta\psi_i + PB\delta u_i\right] dt\\ &\quad + \sum_{i=1}^N \sum_{j=1}^N \delta\beta_i^j dW_j,\ \delta\psi_i(T) = 0.\end{aligned}$$

By Itô's formula, we have

$$\begin{aligned}&\mathbb{E}[\grave{x}_i^T(T) H\delta\grave{x}_i(T)]\\ &= \mathbb{E}[k_i^T(T)\delta\grave{x}_i(T) - k_i^T(0)\delta\grave{x}_i(0)]\\ &= \mathbb{E}\int_0^T \{-[Q\grave{x}_i - (\Psi\bar{x} + \bar{\eta})\\ &\quad - PR_2^{-1}(P\bar{x} + \bar{s}) + \bar{G}^T v]^T \delta\grave{x}_i + k_i^T B\delta u_i\} dt\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}[l^T(T)\delta\psi_i(T) - l^T(0)\delta\psi_i(0)]\\ &= \mathbb{E}\int_0^T \left[(R_2^{-1}v + R_2^{-1}(P\bar{x} + \bar{s}))^T \delta\psi_i - l^T PB\delta u_i\right] dt.\end{aligned}$$

This and (31) gives

$$0 = \mathbb{E}\int_0^T (R_1 u_i + B^T k_i - B^T P l)^T \delta u_i dt$$

which implies $\hat{u}_i = R_1^{-1} B^T (Pl - k_i)$, $1 \leq i \leq N$. ∎

Let $k_i = K\grave{x}_i + \varphi$. Then, by (23) and (30)

$$\begin{aligned}dk_i &= K(A\grave{x}_i - BR_1^{-1}B^T(K\grave{x}_i - Pl + \varphi) + G\bar{x}\\ &\quad - R_2^{-1}(P\bar{x} + \bar{s}))dt + K\sigma dW_i + \dot{K}\grave{x}_i + \dot{\varphi}\\ &= -\left\{A^T(K\grave{x}_i + \varphi) + Q\grave{x}_i - (\Psi\bar{x} + \bar{\eta})\right.\\ &\quad \left. - PR_2^{-1}(P\bar{x} + \bar{s}) + \bar{G}^T v\right\} + \zeta_i dW_i\end{aligned}$$

which implies

$$\begin{aligned}&\dot{K} + A^T K + KA - KBR_1^{-1}B^T K + Q = 0\\ &K(T) = H\end{aligned} \quad (32)$$

$$\begin{aligned}&\dot{\varphi} + (A - BR_1^{-1}B^T K)^T \varphi + KBR_1 B^T Pl\\ &+ K\bar{G}\bar{x} - KR_2^{-1}\bar{s} - (\Psi\bar{x} + \bar{\eta})\\ &- PR_2^{-1}(P\bar{x} + \bar{s}) + \bar{G}^T v = 0,\ \varphi(T) = 0.\end{aligned} \quad (33)$$

Besides, applying (28) into (23), we obtain

$$d\hat{x}^{(N)} = [A\hat{x}^{(N)} - BR_1^{-1}B^T(K\hat{x}^{(N)} - Pl + \varphi) + \bar{G}\bar{x} - R_2^{-1}\bar{s}]dt + \frac{1}{N}\sum_{i=1}^N \sigma dW_i$$

where $\hat{x}^{(N)}(0) = \frac{1}{N}\sum_{i=1}^N x_{i0}$. As an approximation, one can obtain

$$\dot{\bar{x}} = (\bar{A} + \bar{G})\bar{x} + BR_1^{-1}B^T(Pl - \varphi) - R_2^{-1}\bar{s},\ \bar{x}(0) = \bar{x}_0 \quad (34)$$

where $\bar{A} \triangleq A - BR_1^{-1}B^T K$. By (26), (27), (29), (33), and (34), we construct the consistency equations

$$\begin{cases}\dot{\bar{x}} = (\bar{A} + \bar{G})\bar{x} + BR_1^{-1}B^T(Pl - \varphi)\\ \qquad - R_2^{-1}\bar{s},\ \bar{x}(0) = \bar{x}_0\\ \dot{l} = (A + \bar{G})l + R_2^{-1}v + R_2^{-1}(P\bar{x} + \bar{s}),\ l(0) = 0\\ \dot{\bar{s}} = -(A + \bar{G})^T \bar{s} + PBR_1^{-1}B^T(K\bar{x} - Pl + \varphi)\\ \qquad - \bar{\eta},\ \bar{s}(T) = 0\\ \dot{\varphi} = -\bar{A}^T\varphi - KBR_1^{-1}B^T Pl - K\bar{G}\bar{x} + KR_2^{-1}\bar{s}\\ \qquad + \Psi\bar{x} + \bar{\eta} + PR_2^{-1}(P\bar{x} + \bar{s}) - \bar{G}^T v, \varphi(T) = 0\\ \dot{v} = -(A + \bar{G})^T v + (\Psi - Q)\bar{x} + \bar{\eta}\\ \qquad + PR_2^{-1}(P\bar{x} + \bar{s}),\ v(T) = H\bar{x}(T).\end{cases} \quad (35)$$

For further analysis, we assume the following.

(A3) (35) admits a unique solution in $C([0,T], \mathbb{R}^{5n})$.

Note that (35) can be taken as an FBSDE without diffusion terms. The condition of contraction mapping in [27, Th. 5.1] holds necessarily. Thus, (35) must admit a unique solution in a small time duration $[T_0, T]$. However, some additional conditions are needed for existence of a (global) solution to (35) in the time duration $[0, T]$. We now give a sufficient condition that ensures (A3).

Let

$$M_{11} = \begin{bmatrix}\bar{A} + \bar{G} & BR_1^{-1}B^T P\\ R_2^{-1}P & A + \bar{G}\end{bmatrix}$$

$$M_{12} = \begin{bmatrix}-R_2^{-1} & -BR_1^{-1}B^T & 0\\ R_2^{-1} & 0 & R_2^{-1}\end{bmatrix}$$

$$M_{21} = \begin{bmatrix}PBR_1^{-1}B^T K & -PBR_1^{-1}B^T P\\ -K\bar{G} + \Psi + PR_2^{-1}P & 0\\ \Psi - Q + PR_2^{-1}P & 0\end{bmatrix}$$

$$M_{22} = \begin{bmatrix}-(A + \bar{G}) & PBR_1^{-1}B^T K & 0\\ (K + P)R_2^{-1} & -\bar{A} & -\bar{G}^T\\ PR_2^{-1} & 0 & -(A + \bar{G})\end{bmatrix}.$$

Then, (35) can be written as

$$\begin{bmatrix}\dot{\bar{x}}\\ \dot{l}\\ \dot{\bar{s}}\\ \dot{\varphi}\\ \dot{v}\end{bmatrix} = \begin{bmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix}\begin{bmatrix}\bar{x}\\ l\\ \bar{s}\\ \varphi\\ v\end{bmatrix} + \begin{bmatrix}0\\ 0\\ -\bar{\eta}\\ \bar{\eta}\\ \bar{\eta}\end{bmatrix}. \quad (36)$$

Proposition II.4: If the Riccati differential equation

$$\dot{Y} = M_{21} + M_{22}Y - YM_{11} - YM_{12}Y$$

$$Y(T)=\begin{bmatrix}0&0&H^T\\0&0&0\end{bmatrix}^T$$

admits a unique solution $Y\in\mathbb{R}^{3n\times 2n}$ in $[0,T]$, then (A3) holds. Furthermore, under the assumption $\bar\eta=0$, if the Riccati differential equation

$$\dot Z=M_{12}+M_{11}Z-ZM_{22}-ZM_{21}Z, Z(0)=0 \quad (37)$$

admits a solution $Z\in\mathbb{R}^{2n\times 3n}$ in $[0,T]$, then (35) admits a solution in $[0,T]$.

Proof: Denote $m=[\bar x^T,l^T]^T$, and $z=[s^T,\varphi^T,v^T]^T$. Let $z=Ym+\alpha$, $\alpha(T)=0$. Then, $Y(T)=[\begin{smallmatrix}0&0&H^T\\0&0&0\end{smallmatrix}]^T$. By (36)

$$\begin{aligned}\dot z&=\dot Ym+Y(M_{11}m+M_{12}z)+\dot\alpha\\&=(\dot Y+YM_{11}+YM_{12}Y)\bar x+YM_{12}\alpha+\dot\alpha\\&=M_{21}\bar x+M_{22}(Y\bar x+\alpha)+[-\bar\eta^T,\bar\eta^T,\bar\eta^T]^T.\end{aligned}$$

Thus, we obtain

$$\dot Y=M_{21}+M_{22}Y-YM_{11}-YM_{12}Y \quad (38)$$

$$\dot\alpha=(M_{22}-YM_{12})\alpha+[-\bar\eta^T,\bar\eta^T,\bar\eta^T]^T \quad (39)$$

where $Y(T)=[\begin{smallmatrix}0&0&H^T\\0&0&0\end{smallmatrix}]^T$ and $\alpha(T)=0$. Since (38) admits a unique solution, then (39) has a unique solution. Applying $z=Ym+\alpha$ into (36), we have

$$\dot m=M_{11}m+M_{12}(Ym+\alpha),\ m(0)=[\bar x_0^T,0]^T$$

which implies (35) admits a unique solution in $[0,T]$.

Denote $Z=[\begin{smallmatrix}Z_{11}&Z_{12}&Z_{13}\\Z_{21}&Z_{22}&Z_{23}\end{smallmatrix}]$. Note that $s(T)=\varphi(T)=0$, and $v(T)=H\bar x(T)$. We have

$$\begin{bmatrix}0&0\\0&0\\H&0\end{bmatrix}\begin{bmatrix}Z_{11}&Z_{12}&Z_{13}\\Z_{21}&Z_{22}&Z_{23}\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\HZ_{11}(T)&HZ_{12}(T)&HZ_{13}(T)\end{bmatrix}\neq I_{3n}.$$

By the modified Radon's Lemma (see, e.g., [1, Th. 3.1.3]), the proposition follows. ■

C. Asymptotic Optimality

For Problem (PF), we may design the following decentralized control:

$$\hat u_i(t)=-R_1^{-1}B^T[K(t)x_i(t)-P(t)l(t)+\varphi(t)] \quad (40)$$

where K,P are given by (32) and (10), respectively, and l and φ are determined by (35). After the control (40) are applied in (13) and (14), we obtain the following state equations:

$$\begin{aligned}d\hat x_i&=[\bar A\hat x_i+\bar G\hat x^{(N)}+BR_1^{-1}B^T(Pl-\varphi)-R_2^{-1}\hat s]dt\\&\quad+\sigma dW_i,\ i=1,\ldots,N. \quad (41)\\d\hat s&=-\Big[(A+\bar G)^T\hat s-PBR_1^{-1}B^T(K\hat x^{(N)}-Pl+\varphi)+\bar\eta\Big]dt\\&\quad+\frac1N\sum_{i=1}^N\sum_{j=1}^N(\beta_i^j-\frac{\sigma}{N})dW_j,\ \hat s(T)=0. \quad (42)\end{aligned}$$

For further analysis, we assume

(**A4**) The Riccati equation admits a solution

$$\begin{aligned}&\dot{\tilde P}+\tilde P(\bar A+\bar G)+(A+\bar G)^T\tilde P-\tilde PR_2^{-1}\tilde P\\&-PBR_1^{-1}B^TK=0,\ \tilde P(T)=0.\end{aligned}$$

Lemma 2.2: Assume that (A0)–(A1), (A2′), (A3)–(A4) hold. For the system (1) and (2), we have

$$\sup_{0\le t\le T}\mathbb{E}\left(\|\hat x^{(N)}-\bar x\|^2+\|\hat s-\bar s\|^2\right)=O(1/N).$$

Proof: It follows by (41) that

$$\begin{aligned}d\hat x^{(N)}&=[(\bar A+\bar G)\hat x^{(N)}+BR_1^{-1}B^T(Pl-\varphi)\\&\quad-R_2^{-1}\hat s]dt+\frac1N\sum_{i=1}^N\sigma dW_i.\end{aligned}$$

Denote $\xi\triangleq\hat x^{(N)}-\bar x$ and $\chi\triangleq\hat s-\bar s$. From the above equation with (35) and (42), we have

$$\begin{aligned}d\xi&=(\bar A+\bar G)\xi dt-R_2^{-1}\chi dt+\frac1N\sum_{i=1}^N\sigma dW_i\\\xi(0)&=\frac1N\sum_{i=1}^Nx_{i0}-\bar x_0 \quad (43)\\d\chi&=-\left[(A+\bar G)^T\chi-PBR_1^{-1}B^TK\xi\right]dt\\&\quad+\frac1N\sum_{i=1}^N\sum_{j=1}^N(\beta_i^j-\frac{\sigma}{N})dW_j,\ \chi(T)=0. \quad (44)\end{aligned}$$

Let $\chi(t)=\tilde P(t)\xi(t)+\psi(t)$, $t\ge0$. By Itô formula

$$\begin{aligned}d\chi&=\dot{\tilde P}\xi+\tilde P\Bigg\{[(\bar A+\bar G)\xi-R_2^{-1}(\tilde P\xi+\psi)]dt\\&\quad+\frac1N\sum_{j=1}^N\sigma dW_j\Bigg\}+\dot\psi\\&=-\Big[(A+\bar G)^T(\tilde P\xi+\psi)+PBR_1^{-1}B^TK\xi\Big]dt\\&\quad+\frac1N\sum_{i=1}^N\sum_{j=1}^N\left(\beta_i^j-\frac{\sigma}{N}\right)dW_j\end{aligned}$$

which gives $\sum_{i=1}^N\beta_i^j=(\tilde P+I)\sigma$, and

$$\begin{aligned}&\dot{\tilde P}+\tilde P(\bar A+\bar G)+(A+\bar G)^T\tilde P-\tilde PR_2^{-1}\tilde P\\&+PBR_1^{-1}B^TK=0,\ \tilde P(T)=0,\\&\dot\psi+(A+\bar G-R_2^{-1}\tilde P)^T\psi=0,\ \psi(T)=0.\end{aligned}$$

From (A4), we have $\tilde P$ is existent and $\psi(t)\equiv0$. Thus

$$\xi(t)=e^{\Upsilon t}\xi(0)+\frac1N\int_0^te^{\Upsilon(t-\mu)}\sum_{i=1}^N\sigma dW_i(\mu)$$

where $\Upsilon=\bar A+\bar G-R_2^{-1}\tilde P$. By (A0), one can obtain

$$\mathbb{E}\|\xi(t)\|^2\le\frac2N\left\|e^{\Upsilon t}\right\|^2\left\{\max_{1\le i\le N}\mathbb{E}\|x_{i0}\|^2+\int_0^t\left\|e^{-\Upsilon\mu}\sigma\right\|^2d\mu\right\}$$

which completes the proof. ■

Lemma 2.3: If $u = (u_1, \ldots, u_N)$ satisfies

$$\sup_{f\in\mathcal{U}_c^{\rm F}} J_{\rm soc}^{\rm F}(u) \le C$$

then there exists C_1 independent of N such that $\mathbb{E}\int_0^T \|u_i\|^2 dt \le C_1$ for all $i = 1, \ldots, N$.

Proof: Let $f = 0$. Since $R_1 > 0$, then $\sup_f J_{\rm soc}^{\rm F}(u) \le C$ implies $\mathbb{E}\int_0^T \|u_i\|^2 dt \le C_1$ for all $i = 1, \ldots, N$. ■

Lemma 2.4: Assume that (A0), (A1), (A2′), (A3), and (A4) hold. Then, there exists a constant C_0 independent of N such that

$$\sup_{f\in\mathcal{U}_c^{\rm F}} \sum_{i=1}^N J_i^{\rm F}(\hat{u}, f) \le NC_0.$$

Proof: Under (A0), (A1), and (A2′), $\check{f} = -R_2^{-1}(P\check{x}^{(N)} + s)$ is a maximizer of $\sum_{i=1}^N J_i^{\rm F}(\hat{u}, f)$, i.e., $\sum_{i=1}^N J_i^{\rm F}(\hat{u}, \check{f}) = \sup_f \sum_{i=1}^N J_i^{\rm F}(\hat{u}, f)$. By Lemma 2.2, we obtain

$$\sup_{0\le t\le T} \mathbb{E}\|\hat{x}^{(N)}\|^2 \sup_{0\le t\le T}\left(2\mathbb{E}\|\hat{x}^{(N)} - \bar{x}\|^2 + 2\|\bar{x}\|^2\right) \le C$$

$$\sup_{0\le t\le T} \mathbb{E}\|s(t)\|^2 \le \sup_{0\le t\le T}\left(2\mathbb{E}\|\bar{s}\|^2 + 2\mathbb{E}\|s - \bar{s}\|^2\right) \le C.$$

Denote $g \triangleq \bar{G}\hat{x}^{(N)} + BR_1^{-1}B^T(Pl - \varphi) - R_2^{-1}s$. Note that $l, \varphi \in C([0, T], \mathbb{R}^n)$. Then, we have $\sup_{0\le t\le T} \mathbb{E}\|g(t)\|^2 \le C$. It follows from (41) that

$$\mathbb{E}\|\hat{x}_i\|^2 \le C + 3C\left\|\mathbb{E}\int_0^T e^{\bar{A}(T-\tau)}d\tau\right\|^2 + 3\mathbb{E}\int_0^T \left\|e^{\bar{A}(T-\tau)}\sigma\right\|^2 d\tau.$$

From this with (40), we have

$$\sum_{i=1}^N \sup_{f\in\mathcal{U}_c^{\rm F}} J_i^{\rm F}(\hat{u}, f) = \frac{1}{2}\sum_{i=1}^N \mathbb{E}\int_0^T \Big\{\left\|\hat{x}_i - \Gamma\hat{x}^{(N)} - \eta\right\|_Q^2 + \|\hat{u}_i\|_{R_1}^2 - \|\check{f}\|_{R_2}^2\Big\} dt \le NC_0.$$

■

Let $\hat{k}_i \triangleq K\hat{x}_i + \varphi$, where φ is given by (35). We have the following approximation result.

Lemma 2.5: Assume that (A0), (A1), (A2′), (A3), and (A4) hold. Then, for problem (PF), we have

$$\sup_{0\le t\le T} \mathbb{E}\|\hat{k}^{(N)} - v\|^2 = O(1/N)$$

where $\hat{k}^{(N)} = \frac{1}{N}\sum_{i=1}^N \hat{k}_i$ and v is given by (35).

Proof: Let $\vartheta = v - K\bar{x} - \varphi$. By (35) and some elementary calculations, we obtain

$$d\vartheta(t) = -A^T\vartheta(t)dt, \quad \vartheta(T) = 0$$

which implies $\vartheta(t) \equiv 0$. This further gives $v = K\bar{x} + \varphi$. By Lemma 2.2, we have

$$\sup_{0\le t\le T} \mathbb{E}\|\hat{k}^{(N)} - v\|^2 = \sup_{0\le t\le T} \mathbb{E}\|K(\hat{x}^{(N)} - \bar{x})\|^2 \le C\sup_{0\le t\le T} \mathbb{E}\|\hat{x}^{(N)} - \bar{x}\|^2 = O(1/N).$$

This completes the proof. ■

We are in a position to state the result of asymptotic optimality of the decentralized control.

Theorem 2.4: Let (A0), (A1), (A2′), (A3), and (A4) hold. Assume that (P2) is convex. For Problem (PFa), the set of control laws $\hat{u} = (\hat{u}_1, \ldots, \hat{u}_N)$ given by (40) has asymptotic robust social optimality, i.e.,

$$\left|\frac{1}{N}J_{\rm soc}^{\rm wo}(\hat{u}) - \frac{1}{N}\inf_{u\in\mathcal{U}_c^{\rm F}} J_{\rm soc}^{\rm wo}(u)\right| = O(\frac{1}{\sqrt{N}}).$$

Proof: See Appendix A. ■

III. Robust Mean Field Social Control Over an Infinite Horizon

In this section, we consider social optimum control in robust mean field model over an infinite horizon. Let

$$\mathcal{U}_{d,i} = \Big\{u_i \mid u_i(t) \in \sigma(x_i(s), 0 \le s \le t) \quad \mathbb{E}\int_0^\infty e^{-\rho t}\|x_i(t)\|^2 dt < \infty\Big\}$$

and

$$J_i(u, f) = \frac{1}{2}\mathbb{E}\int_0^\infty e^{-\rho t}\Big\{\left\|x_i(t) - \Gamma x^{(N)}(t) - \eta\right\|_Q^2 + \|u_i(t)\|_{R_1}^2 - \|f(t)\|_{R_2}^2\Big\} dt \tag{45}$$

where $\rho \ge 0$.

Problem (PIa): Seek a set of decentralized control to asymptotically optimize the social cost under the worst-case disturbance for System (1) and (45), where $J_{\rm soc}(u, f) = \sum_{i=1}^N J_i(u, f)$, and

$$\mathcal{U}_c = \Big\{u_i| \; u_i(t) \in \sigma(x_i(0), W_i(s), 0 \le s \le t, \; 1 \le i \le N) \quad \mathbb{E}\int_0^\infty e^{-\rho t}\|x_i(t)\|^2 dt < \infty\Big\}.$$

A. Decentralized Control Design

Let $u_i = \check{u}_i \in \mathcal{U}_c, i = 1, \ldots, N$ be fixed. The optimal control problem with respect to drift uncertainty is as follows:

(P4) minimize$_{f\in\mathcal{U}_c}\check{J}_{\rm soc}(\check{u}, f)$

where

$$\check{J}_{\rm soc}(\check{u}, f) = \frac{1}{2}\sum_{i=1}^N \mathbb{E}\int_0^\infty e^{-\rho t}\Big\{-\left\|x_i(t) - \Gamma x^{(N)}(t) - \eta\right\|_Q^2 + \|f(t)\|_{R_2}^2\Big\} dt. \tag{46}$$

1) An Example of a Scalar Model: Consider the case of uniform agents with scalar states. Let $A = a, \eta = 0, G = 0, Q = 1, \Gamma = \gamma, \; R_2 = r_2, \; x_i \in \mathbb{R}$ and $\check{u}_i = 0, i = 1, \ldots, N$. By rearranging the integrand of $\check{J}_{\rm soc}$, we have

$$\check{J}_{\rm soc} = \frac{1}{2}\mathbb{E}\int_0^\infty e^{-\rho t}\left(-x^T\hat{Q}x + Nr_2f^2\right) dt$$

where $x = (x_1, \ldots, x_N)^T$, and $\hat{Q} = (\hat{q}_{ij})$ is given by

$$\hat{q}_{ii} = 1 + (\gamma^2 - 2\gamma)/N, \quad \hat{q}_{ij} = (\gamma^2 - 2\gamma)/N, \; i \ne j.$$

Introduce the Riccati equation

$$2\left(a - \frac{\rho}{2}\right)P - \frac{1}{Nr_2}P\mathbf{1}\mathbf{1}^TP - \hat{Q} = 0. \tag{47}$$

By observation, P has the form

$$p_{ij} = \begin{cases} p & \text{if } i=j \\ q & \text{if } i \neq j. \end{cases}$$

Denote $\bar{a} = a - \frac{\rho}{2}$. By solving (47), we obtain the maximal solution as follows: $p = q + \frac{1}{2\bar{a}}$ and

$$q = \frac{1}{N}\left(r_2\bar{a} - \frac{1}{2\bar{a}} + \sqrt{r_2^2\bar{a}^2 + r_2(\gamma^2 - 2\gamma - 1)}\right).$$

The optimal control is given by

$$\check{f} = \frac{1}{Nr_2}\mathbf{1}^T P x = \left[\bar{a} + \sqrt{\bar{a}^2 + \frac{1}{r_2}(\gamma^2 - 2\gamma - 1)}\right] x^{(N)}.$$

■

For general systems, we make the following assumptions.

(A5) Problem (P4) is uniformly convex in f;

(A6) $A + G - \frac{\rho}{2}I$ is Hurwitz.

Below are some equivalent conditions to (A5).

Proposition 3.1: Let (A0) and (A6) hold. Then, (A5) holds, i.e., (P4) is uniformly convex, if and only if one of (i)–(iv) holds.

i) For any $f \in \mathcal{U}_c$, there exists $\delta > 0$ such that

$$J'_{\rm soc}(f) = \frac{1}{2}\mathbb{E}\int_0^\infty e^{-\rho t}\left(-\mathbf{y}^T\hat{\mathbf{Q}}\mathbf{y} + Nf^T R_2 f\right) dt \geq \delta N\mathbb{E}\int_0^\infty e^{-\rho t}\|f\|^2 dt$$

where $\mathbf{y} \in \mathbb{R}^{nN}$ satisfies

$$d\mathbf{y} = (\check{\mathbf{A}}\mathbf{y} + \mathbf{1}\otimes f)dt,\ \mathbf{y}(0) = 0.$$

ii) The equation

$$\rho\mathbf{P} = \check{\mathbf{A}}^T\mathbf{P} + \mathbf{P}\check{\mathbf{A}} - \hat{\mathbf{Q}} - \mathbf{P}(\mathbf{1}\otimes I)(NR_2)^{-1}(\mathbf{1}^T\otimes I)\mathbf{P}$$

admits a solution such that $\check{\mathbf{A}} - (\mathbf{1}\otimes I)(NR_2)^{-1}(\mathbf{1}^T\otimes I)\mathbf{P} - \frac{\rho}{2}(I_N \otimes I)$ is Hurwitz.

iii) The equation

$$\rho P = (A+G)^T P + P(A+G) - PR_2^{-1}P + \Psi - Q$$

admits a solution such that $A + \bar{G} - \frac{\rho}{2}I$ is Hurwitz.

iv) The real part of any eigenvalue of M is not zero, where

$$M = \begin{bmatrix} A + G - \frac{\rho}{2}I & R_2^{-1} \\ -(I-\Gamma)^T Q(I-\Gamma) & -A^T - G^T + \frac{\rho}{2}I \end{bmatrix}.$$

Proof: (A5)⇔(i) follows by [25, Lemma 1]. We now prove (ii)⇒(A5). If (ii) holds, then by the completion of squares technique, we can obtain

$$J'_{\rm soc}(f) = \mathbb{E}\int_0^\infty e^{-\rho t}N\left\|f(t) + \frac{1}{N}R_2^{-1}(\mathbf{1}^T\otimes I)\mathbf{P}\mathbf{y}(t)\right\|^2_{R_2} dt \geq 0.$$

Clearly, $J'_{\rm soc}(f) = 0$ leads to $f(t) = -\frac{1}{N}R_2^{-1}(\mathbf{1}^T\otimes I)\mathbf{P}\mathbf{y}(t)$, which together with $\mathbf{y}(0) = 0$ further implies $f(t) \equiv 0$. From [15], we obtain that $J'_{\rm soc}(f)$ is positive definite, which implies that (P4) is uniformly convex. Note that $(\check{\mathbf{A}} - \frac{\rho}{2}(I_N \otimes I)), I_N \otimes I)$ is stabilizable. From (5) and (A5)⇔(i), (P4) is uniformly convex if and only if there exists $\delta > 0$ such that

$$\int_0^\infty e^{-\rho t}\left(-\|(I-\Gamma)y_i\|_Q^2 + f^T R_2 f\right) dt \geq \int_0^\infty e^{-\rho t}\delta\|f\|^2 dt.$$

Following the proof of (ii)⇒(A5), we obtain (iii)⇒(A5). Since (A6) holds, it follows by [34, Th. 5.3] that (A5)⇒(iii). Note that $(\mathbf{1}^T\otimes I)\check{\mathbf{A}} = \mathbf{1}^T\otimes(A+G)$, $\check{\mathbf{A}}(\mathbf{1}\otimes I) = \mathbf{1}\otimes(A+G)$, and $\frac{1}{N}(\mathbf{1}^T\otimes I)\hat{\mathbf{Q}}(\mathbf{1}^T\otimes I) = Q - \Psi$. We have $\frac{1}{N}(\mathbf{1}^T\otimes I)\mathbf{P}(\mathbf{1}^T\otimes I) = P$. From (4) and $y_1 = y_2 = \cdots = y^{(N)}$, we obtain (iii)⇔(ii). (iii)⇔(iv) is implied from [28]. ■

Remark 3.1: From the proof of Proposition 3.1, Assumption (A5) implies $J'_{soc}(f) \geq 0$, i.e., (P4) is convex in f.

With some abuse of notation, in this section, we still use $P, K, Y, Z, s, \bar{s}, \varphi, v \ldots$. But here P, K, Y, Z are time-invariant and $s, \bar{s}, \varphi, v$ are functions of time $t \in [0,\infty)$. Following (10)–(12), we may construct $\check{f} = -R_2^{-1}(Px^{(N)} + \check{s})$, where $P \in \mathbb{R}^{n\times n}$ and $\check{s} \in L^2_{\mathcal{F},\frac{\rho}{2}}(0,\infty;\mathbb{R}^n)$ are determined by

$$\left(A + G - \frac{\rho}{2}I\right)^T P + P\left(A + G - \frac{\rho}{2}I\right) - PR_2^{-1}P - (I-\Gamma)^T Q(I-\Gamma) = 0, \tag{48}$$

$$d\check{s} + [(A + G - R_2^{-1}P - \rho I)^T\check{s} + PB\check{u}^{(N)} + \bar{\eta}]dt + \frac{1}{N}\sum_{i=1}^N \zeta_i dW_i = 0. \tag{49}$$

Theorem 3.1: Under (A0), (A1), and (A5), Problem (P4) has a minimizer $\check{f} = -R_2^{-1}(Px^{(N)} + \check{s})$, where P is the maximal solution of (48) and $\check{s}$ is the unique solution of (49) in $L^2_{\mathcal{F},\frac{\rho}{2}}(0,\infty;\mathbb{R}^n)$.

Proof: Denote $\acute{x}_i = e^{-\frac{\rho}{2}t}x_i$, $\acute{u}_i = e^{-\frac{\rho}{2}t}u_i$, and $\acute{f} = e^{-\frac{\rho}{2}t}f$. It follows by (1) and (46) that

$$d\acute{x}_i(t) = \left[\left(A - \frac{\rho}{2}I\right)\acute{x}_i(t) + B\acute{u}_i(t) + G\acute{x}^{(N)}(t) + \acute{f}(t)\right]dt + e^{-\frac{\rho}{2}t}\sigma dW_i(t),\quad 1 \leq i \leq N$$

$$\check{J}_{\rm soc}(\check{u}, f) = \frac{1}{2}\sum_{i=1}^N \mathbb{E}\int_0^\infty \left\{-\left\|\acute{x}_i(t) - \Gamma\acute{x}^{(N)}(t) - e^{-\frac{\rho}{2}t}\eta\right\|_Q^2 + \|\acute{f}(t)\|^2_{R_2}\right\} dt.$$

By a similar argument in the proof of Theorem 2.1, we obtain $\check{J}_{\rm soc}(\check{u}, f) = \varepsilon\Lambda'_1 + \frac{\varepsilon^2}{2}\Lambda'_2$, where

$$\Lambda'_1 \triangleq \sum_{i=1}^N \mathbb{E}\int_0^\infty \left[\left\langle -Q\left(\acute{x}_i - (\Gamma\acute{x}^{(N)} + \eta)\right), \acute{y}_i - \Gamma\acute{y}^{(N)}\right\rangle + \langle R_2\acute{f}, \acute{f}\rangle\right] dt$$

$$\Lambda'_2 \triangleq \sum_{i=1}^N \mathbb{E}\int_0^\infty \left\{-\left\|\acute{y}_i - \Gamma\acute{y}^{(N)}\right\|_Q^2 + \|\acute{f}\|^2_{R_2}\right\} dt$$

and $\acute{y}_i$ satisfies

$$d\acute{y}_i = \left[\left(A - \frac{\rho}{2}\right)\acute{y}_i + G\acute{y}^{(N)} + \acute{f}\right]dt,\quad \acute{y}_i(0) = 0.$$

By (A5), $\Lambda'_2 \geq 0$. Problem (P4) has a unique minimizer $\acute{f} = -R_2^{-1}\acute{p}^{(N)}$ with $\acute{p}^{(N)} = \frac{1}{N}\sum_{i=1}^N \acute{p}_i$ if and only if

$$\begin{cases} d\acute{x}_i = \left[\left(A - \frac{\rho}{2}I\right)\acute{x}_i + B\acute{u}_i - R_2^{-1}\acute{p}^{(N)} + G\acute{x}^{(N)}\right]dt + e^{-\rho t}\sigma dW_i,\ x_i(0) = x_{i0} \\ d\acute{p}_i = -\left[\left(A - \frac{\rho}{2}I\right)^T\acute{p}_i + G^T\acute{p}^{(N)} - Q\acute{x}_i + \Psi\acute{x}^{(N)} + e^{-\rho t}\bar{\eta}\right]dt + \sum_{j=1}^N \acute{\beta}_i^j dW_j \end{cases} \tag{50}$$

admits a set of solutions $(\acute{x}_i, \acute{p}_i, i = 1, \ldots, N)$ in $L^2_{\mathcal{F},\frac{\rho}{2}}(0,\infty;\mathbb{R}^n)$. It follows from (50) that

$$\begin{cases} d\acute{x}^{(N)} = \big[\big(A + G - \frac{\rho}{2}I\big)\acute{x}^{(N)} + B\acute{u}^{(N)} \\ \qquad -R_2^{-1}\acute{p}^{(N)}\big]\,dt + e^{-\rho t}\sigma dW_i,\, x_i(0) = x_{i0} \\ d\acute{p}^{(N)} = -\Big[\big(A + G - \frac{\rho}{2}I\big)^T\acute{p}^{(N)} + (\Psi - Q)\acute{x}^{(N)} \\ \qquad +e^{-\rho t}\bar{\eta}\big]\,dt + \sum_{j=1}^N \acute{\beta}_i^j dW_j. \end{cases} \tag{51}$$

Note that $A + G - \frac{\rho}{2}I$ is Hurwitz. By (A5) and Proposition 3.1, we obtain that (48) admits a maximal solution such that $A + G - \frac{\rho}{2}I - R_2^{-1}P$ is Hurwitz, which with [34, Lemma 2.5] gives that (49) admits a unique solution in $L^2_{\mathcal{F},\frac{\rho}{2}}(0,\infty;\mathbb{R}^n)$. Let $\acute{p}^{(N)} = P\acute{x}^{(N)} + \acute{s}$, where $\acute{s} = e^{-\frac{\rho}{2}t}\check{s}$. Then, we have that $(\acute{x}^{(N)}, \acute{p}^{(N)})$ is a solution of (51). By a similar argument to Theorem 2.2, the proof is completed. ■

After the worst-case drift $\check{f}$ is applied, we have the following optimal control problem.

(P2′): Minimize $J_{\rm soc}(u, \check{f}(u))$ over $\{u_i, 1 \le i \le N)|u_i \in \mathcal{U}_c\}$, where $s \in L^2_{\mathcal{F},\frac{\rho}{2}}(0,\infty;\mathbb{R}^n)$

$$dx_i = [Ax_i + Bu_i + Gx^{(N)} - R_2^{-1}(Px^{(N)} + s)]dt \\ + \sigma dW_i, x_i(0) = x_{i0}, 1 \le i \le N \tag{52}$$

$$ds = -\Big[(A + \bar{G})^T s + PBu^{(N)} + \bar{\eta}\Big]\,dt \\ + \frac{1}{N}\sum_{i=1}^N\sum_{j=1}^N\Big(\beta_i^j - \frac{\sigma}{N}\Big)\,dW_j \tag{53}$$

$$J_{\rm soc}(u) = \frac{1}{2}\sum_{i=1}^N \mathbb{E}\int_0^\infty e^{-\rho t}\Big\{\big\|x_i - \Gamma x^{(N)} - \eta\big\|_Q^2 \\ + \|u_i\|_{R_1}^2 - \|Px^{(N)} + s\|^2_{R_2^{-1}}\Big\}\,dt. \tag{54}$$

Lemma 3.1: Assume that (A0), (A5), and (A6) hold, and there exists a constant $C_0' > 0$ such that $R_1 > C_0'I$ and $R_2 > C_0'I$. Then, Problem (P2′) is uniformly convex.

Proof: Let $\mathbf{z} \in \mathbb{R}^{nN}$ and $\grave{s} \in \mathbb{R}^n$ satisfy

$$d\mathbf{z} = (\bar{\mathbf{A}}\mathbf{z} + \mathbf{B}\mathbf{u} - \mathbf{1} \otimes R_2^{-1}\grave{s})dt,\ \mathbf{z}(0) = 0 \tag{55}$$

$$d\grave{s} = -\Big[(A + \bar{G})^T\grave{s} + \frac{1}{N}PB(\mathbf{1}^T \otimes I)\mathbf{u}\Big]\,dt \\ + \frac{1}{N}\sum_{i=1}^N\sum_{j=1}^N \grave{\beta}_i^j dW_j. \tag{56}$$

By a similar argument with [25], we obtain that Problem (P2) is uniformly convex if for any $u_i \in \mathcal{U}_c$, there exists $\delta > 0$ such that

$$\mathbb{E}\int_0^\infty e^{-\rho t}\left(\mathbf{z}^T\bar{\mathbf{Q}}\mathbf{z} + \mathbf{u}^T\mathbf{R}_1\mathbf{u} - N\grave{s}^T R_2^{-1}\grave{s}\right)dt \\ \ge \delta\mathbb{E}\int_0^\infty e^{-\rho t}\|\mathbf{u}\|^2 dt. \tag{57}$$

Note that $A + \bar{G} - \frac{\rho}{2}I$ is Hurwitz. By [34, Lemma 2.5] and (56)

$$\mathbb{E}\int_0^\infty e^{-\rho t}\|\acute{s}(t)\|^2 dt \le \frac{C_1}{N}\mathbb{E}\int_0^\infty e^{-\rho t}\|\mathbf{u}(t)\|^2 dt. \tag{58}$$

Since $\bar{\mathbf{A}} - \frac{\rho}{2}I$ is Hurwitz, from (55) and (58), we obtain

$$\mathbb{E}\int_0^\infty e^{-\rho t}\|\mathbf{z}\|^2 dt \le C\int_0^\infty e^{-\rho t}\mathbb{E}\|\mathbf{u}\|^2 dt.$$

Note that $\lambda_{\min}(\bar{\mathbf{Q}}) \ge -\lambda_{\max}(\Psi + PR_2^{-1}P)$. Thus, there exists $C_0' > 0$ such that for $R_1 \ge C_0'I$ and $R_2 \ge C_0'I$, (57) holds. ■

Based on the analysis in Section II-B, we construct an auxiliary optimal control problem.

(P5): Minimize $\bar{J}_i(u_i)$ over $u_i \in \mathcal{U}_i$, where

$$d\grave{x}_i = [A\grave{x}_i + Bu_i + G\bar{x} - R_2^{-1}(P\bar{x} + \bar{s})]dt + \sigma dW_i \\ \grave{x}_i(0) = x_{i0} \\ d\psi_i = -\big[(A + \bar{G})^T\psi_i + PBu_i\big]\,dt + z_i^i dW_i$$

$$\bar{J}_i(u_i) = \frac{1}{2}\mathbb{E}\int_0^\infty e^{-\rho t}\big[\grave{x}_i^T Q\grave{x}_i - 2(\Psi\bar{x} + \bar{\eta})^T\grave{x}_i \\ - 2(P\bar{x} + \bar{s})^T R_2^{-1}P\grave{x}_i + 2v^T(\bar{G}\grave{x}_i - R_2^{-1}\psi_i) \\ -2(P\bar{x} + \bar{s})^T R_2^{-1}\psi_i + u_i^T R_1 u_i\big]\,dt.$$

Here $\bar{s}, v \in C_{\rho/2}([0,\infty),\mathbb{R}^n)$ are determined by

$$\dot{\bar{s}} = -(A + \bar{G} - \rho I)^T\bar{s} - PB\bar{u} - \bar{\eta} \\ \dot{v} = -(A + \bar{G} - \rho I)^T v + (\Psi - Q)\bar{x} + \bar{\eta} + PR_2^{-1}(P\bar{x} + s).$$

By using the method in [38] and [44], we can show that if (A6) holds and $Q \ge 0$, (P5) admits the unique optimal control

$$\hat{u}_i(t) = -R_1^{-1}B^T(Kx_i(t) - Pl(t) + \varphi(t)) \tag{59}$$

where $K \in \mathbb{R}^{n\times n}$ and $l, \varphi \in C_{\rho/2}([0,\infty),\mathbb{R}^n)$ are determined by

$$\rho K = A^T K + KA - KBR_1^{-1}B^T K + Q \\ \dot{l} = (A + \bar{G})l + R_2^{-1}v + R_2^{-1}(P\bar{x} + \bar{s}),\ l(0) = 0 \\ \rho\varphi = \dot{\varphi} + \bar{A}^T\varphi + KBR_1^{-1}B^T Pl + K\bar{G}\bar{x} \\ - KR_2^{-1}\bar{s} - (\Psi\bar{x} + \bar{\eta}) - PR_2^{-1}(P\bar{x} + \bar{s}) + \bar{G}^T v.$$

By applying the control (59) into (52) combined with mean field approximations, we obtain the following equation system:

$$\begin{cases} \dot{\bar{x}} = (\bar{A} + \bar{G})\bar{x} + BR_1^{-1}B^T(Pl - \varphi) - R_2^{-1}\bar{s}, \bar{x}(0) = \bar{x}_0 \\ \dot{l} = (A + \bar{G})l + R_2^{-1}v + R_2^{-1}(P\bar{x} + \bar{s}),\ l(0) = 0 \\ \dot{\bar{s}} = -(A + \bar{G} - \rho I)^T\bar{s} + PBR_1^{-1}B^T(K\bar{x} + \varphi) - \bar{\eta} \\ \dot{\varphi} = -(\bar{A} - \rho I)^T\varphi - KBR_1^{-1}B^T Pl - K\bar{G}\bar{x} \\ \qquad + KR_2^{-1}\bar{s} + \Psi\bar{x} + \bar{\eta} + PR_2^{-1}(P\bar{x} + \bar{s}) - \bar{G}^T v \\ \dot{v} = -(A + \bar{G} - \rho I)^T v + (\Psi - Q)\bar{x} + \bar{\eta} \\ \qquad + PR_2^{-1}(P\bar{x} + \bar{s}). \end{cases} \tag{60}$$

For further analysis, we assume.

(A7) (60) admits a unique solution $(\bar{x}, l, s, \varphi, v)$ in $C_{\rho/2}([0,\infty),\mathbb{R}^{5n})$.

The existence and uniqueness of a solution to (60) may be obtained by using fixed-point methods similar to those in [18] and [38]. We now give a sufficient condition that ensures (A7) by virtue of Riccati equations. Using the notation in Section II-B, we have

$$\begin{bmatrix}\dot{\bar{x}}\\ \dot{l}\\ \dot{s}\\ \dot{\varphi}\\ \dot{v}\end{bmatrix}=\begin{bmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}+\rho I_{3n}\end{bmatrix}\begin{bmatrix}\bar{x}\\ l\\ s\\ \varphi\\ v\end{bmatrix}+\begin{bmatrix}0\\ 0\\ -\bar{\eta}\\ \bar{\eta}\\ \bar{\eta}\end{bmatrix}. \tag{61}$$

Proposition 3.2: If the algebraic Riccati equation

$$M_{21}+\rho Y+M_{22}Y-YM_{11}-YM_{12}Y=0$$

admits a solution $Y\in\mathbb{R}^{3n\times 2n}$ such that both $M_{11}+M_{12}Y-\frac{\rho}{2}I_{2n}$ and $-M_{22}+YM_{12}-\frac{\rho}{2}I_{3n}$ are Hurwitz, then (A7) holds.

Proof: Denote $m=[\bar{x}^T,l^T]^T$, $z=[s^T,\varphi^T,v^T]^T$. Let $z=Ym+\alpha$. By (61) and Itô's formula, we obtain

$$0=M_{21}+(M_{22}+\rho I_{3n})Y-YM_{11}-YM_{12}Y \tag{62}$$

$$\dot{\alpha}=(M_{22}+\rho I_{3n}-YM_{12})\alpha+[-\bar{\eta}^T,\bar{\eta}^T,\bar{\eta}^T]^T. \tag{63}$$

Since (62) admits a solution such that $-M_{22}+YM_{12}-\frac{\rho}{2}I_{3n}$ is Hurwitz, then (63) has a unique solution

$$\alpha(t)=-\int_t^\infty \exp\left[(YM_{12}-M_{22}-\rho I_{3n})(\tau-t)\right]\cdot[-\bar{\eta}^T,\bar{\eta}^T,\bar{\eta}^T]^T d\tau.$$

Applying $z=Ym+\alpha$ into (61), we have

$$\dot{m}=(M_{11}+M_{12}Y)m+M_{12}\alpha.$$

Since $M_{11}+M_{12}Y-\frac{\rho}{2}I_{2n}$ is Hurwitz, then $[\bar{x}^T,l^T]^T\in C_{\rho/2}([0,\infty),\mathbb{R}^{2n})$, and this further implies that (60) admits a unique solution in $C_{\rho/2}([0,\infty),\mathbb{R}^{5n})$. ∎

B. Asymptotic Optimality

Let

$$\hat{u}_i(t)=-R_1^{-1}B^T(Kx_i(t)-Pl(t)+\varphi(t)) \tag{64}$$

where l and φ are determined by (60). After the control $\hat{u}_i$ is applied, the closed-loop dynamics can be written as

$$d\hat{x}_i=[\bar{A}\hat{x}_i+\bar{G}\hat{x}^{(N)}+BR_1^{-1}B^T(Pl-\varphi)-R_2^{-1}\hat{s}]dt+\sigma dW_i.$$

For further analysis, we assume

(A8) The equation

$$\tilde{P}(\bar{A}+\bar{G})+(A+\bar{G})^T\tilde{P}-\tilde{P}R_2^{-1}\tilde{P}+PBR_1^{-1}B^TK=0 \tag{65}$$

admits a solution $\tilde{P}$ such that $\bar{A}+\bar{G}-\frac{\rho}{2}I-R_2^{-1}\tilde{P}$ and $A+\bar{G}-\frac{\rho}{2}I-R_2^{-1}\tilde{P}$ are Hurwitz, where $\bar{A}=A-BR_1^{-1}B^TK$ and $\bar{G}=G-R_2^{-1}P$.

Theorem 3.2: Assume (i) (A0), (A1), (A5)–(A8) hold, (ii) $A-\frac{\rho}{2}I$ is Hurwitz , and (iii) (P2′) is convex. For Problem (PIa), the set of control laws $\hat{u}=(\hat{u}_1,\ldots,\hat{u}_N)$ given by (40) has asymptotic robust social optimality, i.e.,

$$\left|\frac{1}{N}\sup_{f\in\mathcal{U}_c}J_{\rm soc}(\hat{u},f)-\frac{1}{N}\inf_{u_i\in\mathcal{U}_c}\sup_{f\in\mathcal{U}_c}J_{\rm soc}(u,f)\right|=O\left(\frac{1}{\sqrt{N}}\right).$$

Proof: See Appendix B. ∎

IV. Numerical Example

We now give a numerical example for Problem **(PF)** to verify the result. Take the parameters $A=B=R_1=R_2=Q=H=1, G=-1.5, \Gamma=0.5, \eta=0$, and $T=1$. By solving (10), we can obtain that $P(t)=-\frac{1}{(t+1)}-\frac{1}{2}$, which is shown in Fig. 1. By Proposition 2.2, (A2′) holds. For (37) in Proposition 2.4, the curves of all entries of the solution Z are given in Fig. 2. It can be seen that when $t\in[0,0.7]$, (37) admits a solution. By MATLAB computation, the solution blows up at $t=0.758276$. From Proposition 2.4, when $t\in[0,0.7]$, (A3) holds. The curve of $\tilde{P}$ is shown in Fig. 3. It can be seen that the Riccati equation in (A4) admits a solution $\tilde{P}$ when $t\in[0,0.8]$. As a conclusion, when $t\in[0,0.7]$, (A0), (A1), (A2′), (A3), and (A4) hold. By Theorem 2.4, Problem (PF) admits a set of control laws, which has asymptotic robust social optimality.

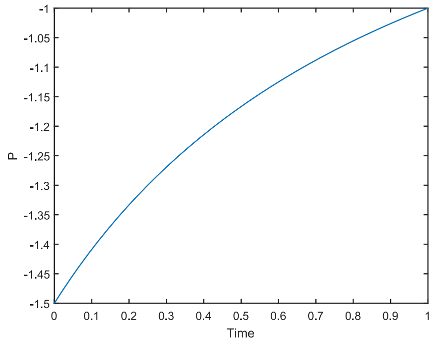

Fig. 1. The curve of $P(t)$.

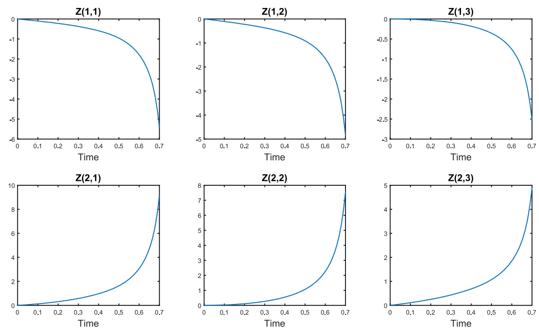

Fig. 2. The curves of all entries of $Z\in\mathbb{R}^{2\times 3}$ when $t\in[0,0.7]$.

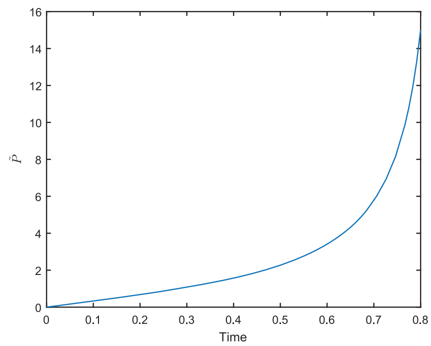

Fig. 3. The curve of $\tilde{P}(t)$.

V. Concluding Remarks

This article considered a class of mean field LQG social optimum problem with global drift uncertainty. Based on the soft control approach, a set of decentralized strategies is designed by optimizing the worst-case cost subject to consistent requirements in mean field approximations. Such set of strategies is further shown to be robust social optimal.

For further work, it is of interest to consider mean field team optimization with volatility-uncertain common noise. Due to common noise and volatility uncertainty, all agents are coupled via some high-dimensional FBSDE systems. Other interesting topics include robust mean-field type social control [8], [30], mean field social control with partial information [39], [43], and with discrete sampling [26], [32].

APPENDIX A
PROOF OF THEOREM 2.4

Proof: Note that we only need to optimize the social cost under worst-case disturbance $J^{\rm wo}_{\rm soc}(u)$. By Theorem 2.2, Problem (P2) is equivalent to (PF). From Lemma 2.4, one can obtain that for (P2)

$$\mathbb{E}\int_0^T(\|\hat{x}_i\|^2+\|\hat{u}_i\|^2)dt<C. \quad \text{(A.1)}$$

It suffices to consider all $u_i\in\mathcal{U}_c^{\rm F}$ such that $J^{\rm wo}_{\rm soc}(u)\le J^{\rm wo}_{\rm soc}(\hat{u})\le NC_0$. By Lemma 2.3

$$\mathbb{E}\int_0^T\|u_i\|^2dt<C,\ i=1,\dots,N \quad \text{(A.2)}$$

which implies

$$\mathbb{E}\int_0^T\|u^{(N)}\|^2dt<C. \quad \text{(A.3)}$$

By (14) and [44, Ch. 7], we have

$$\mathbb{E}\int_0^T\|s\|^2dt\le C_1\mathbb{E}\int_0^T\|u^{(N)}\|^2dt<C.$$

From (13)

$$dx^{(N)}=\left[(A+\bar{G})x^{(N)}+Bu^{(N)}-R_2^{-1}s\right]dt+\frac{1}{N}\sum_{i=1}^N\sigma dW_i$$

which together with (A.3) implies $\mathbb{E}\int_0^T\|x^{(N)}\|^2dt<C$. This with (A.2) leads to

$$\mathbb{E}\int_0^T(\|x_i\|^2+\|u_i\|^2+\|s\|^2)dt<C. \quad \text{(A.4)}$$

Let $\tilde{x}_i=x_i-\hat{x}_i$, $\tilde{u}_i=u_i-\hat{u}_i$, $i=1,\dots,N$, $\tilde{x}^{(N)}=\frac{1}{N}\sum_{j=1}^N\tilde{x}_j$ and $\tilde{s}=s-\hat{s}$. Then, by (13)

$$d\tilde{x}_i=(A\tilde{x}_i+\bar{G}\tilde{x}^{(N)}+B\tilde{u}_i-R_2^{-1}\tilde{s})dt,\ \tilde{x}_i(0)=0. \quad \text{(A.5)}$$

$$d\tilde{s}=-\left[(A+\bar{G})^T\tilde{s}+PB\tilde{u}^{(N)}\right]dt+\frac{1}{N}\sum_{i=1}^N\sum_{j=1}^N\tilde{\beta}_i^jdW_j,\ \tilde{s}(T)=0. \quad \text{(A.6)}$$

By (A.1) and (A.4)

$$\mathbb{E}\int_0^T(\|\tilde{x}_i\|^2+\|\tilde{u}_i\|^2+\|\tilde{s}\|^2)dt<C.$$

From (15), we have

$$\begin{aligned}J^{\rm F}_{\rm soc}(u)=&\frac{1}{2}\sum_{i=1}^N\mathbb{E}\int_0^T\Big[\big\|\hat{x}_i-\Gamma\hat{x}^{(N)}-\eta+\tilde{x}_i-\Gamma\tilde{x}^{(N)}\big\|^2_Q+\big\|\hat{u}_i+\tilde{u}_i\big\|^2_{R_1}\\&-\big\|P(\hat{x}^{(N)}+\tilde{x}^{(N)})+\hat{s}+\tilde{s}\big\|^2_{R_2^{-1}}\Big]dt+\frac{1}{2}\mathbb{E}\big\|\hat{x}_i(T)+\tilde{x}_i(T)\big\|^2_H\\=&\sum_{i=1}^N(J_i^{\rm F}(\hat{u})+\tilde{J}_i^{\rm F}(\tilde{u})+I_i)\end{aligned} \quad \text{(A.7)}$$

where

$$\begin{aligned}\tilde{J}_i^{\rm F}(\tilde{u})\triangleq&\frac{1}{2}\mathbb{E}\int_0^T\Big[\|\tilde{x}_i-\Gamma\tilde{x}^{(N)}\|^2_Q+\|\tilde{u}_i\|^2_{R_1}\\&-\|P\tilde{x}^{(N)}\|^2_{R_2^{-1}}\Big]dt+\frac{1}{2}\mathbb{E}\big\|\tilde{x}_i(T)\big\|^2_H\end{aligned}$$

$$\begin{aligned}I_i=&\mathbb{E}\int_0^T\Big[\left(\hat{x}_i-\Gamma\hat{x}^{(N)}-\eta\right)^TQ\left(\tilde{x}_i-\Gamma\tilde{x}^{(N)}\right)\\&+\hat{u}_i^TR_1\tilde{u}_i-\left(P(\hat{x}^{(N)}+\hat{s})\right)^TR_2^{-1}\\&\times(P\tilde{x}^{(N)}+\tilde{s})\Big]dt+\mathbb{E}[\hat{x}_i^T(T)H\tilde{x}_i(T)].\end{aligned}$$

By Lemma 2.1, Problem (P2) is uniformly convex. As shown in [15] and [25], for any $\lambda_1\in(0,1)$ and $\lambda_2=1-\lambda_1$, we have

$$\begin{aligned}&\lambda_1\lambda_2\sum_{i=1}^N\tilde{J}_i^{\rm F}(\tilde{u})\\&=\lambda_1J^{\rm F}_{\rm soc}(u)+\lambda_2J^{\rm F}_{\rm soc}(\hat{u})-J^{\rm F}_{\rm soc}(\lambda_1u+\lambda_2\hat{u})\ge0\end{aligned}$$

which implies $\sum_{i=1}^N\tilde{J}_i^{\rm F}(\tilde{u})\ge0$. We now prove $\frac{1}{N}\sum_{i=1}^NI_i=O(\frac{1}{\sqrt{N}})$. By straightforward computation

$$\begin{aligned}\sum_{i=1}^NI_i=&\sum_{i=1}^N\mathbb{E}\int_0^T\Big\{\tilde{x}_i^T\Big[Q(\hat{x}_i-\Gamma\hat{x}^{(N)}-\eta)\\&-\Gamma^TQ((I-\Gamma)\hat{x}^{(N)}-\eta)\Big]+\hat{u}_i^TR_1\tilde{u}_i\Big\}dt\\&-N\mathbb{E}\int_0^T\left(P\hat{x}^{(N)}+\hat{s}\right)^TR_2^{-1}(P\tilde{x}^{(N)}+\tilde{s})dt\\&+\sum_{i=1}^N\mathbb{E}[\hat{x}_i^T(T)H\tilde{x}_i(T)]\\=&\sum_{i=1}^N\mathbb{E}\int_0^T\{\tilde{x}_i^T(Q\hat{x}_i-\Psi\bar{x}-\eta)+\hat{u}_i^TR_1\tilde{u}_i\\&-(P\bar{x}+\hat{s})^TR_2^{-1}P\tilde{x}_i\Big\}dt\\&+\sum_{i=1}^N\mathbb{E}\int_0^T\xi^T\left[(\Psi-PR_2^{-1}P)\tilde{x}_i-PR_2^{-1}\tilde{s}\right]dt\\&-N\mathbb{E}\int_0^T(P\bar{x}+\hat{s})^TR_2^{-1}\tilde{s}dt+\sum_{i=1}^N\mathbb{E}\|\hat{x}_i^T(T)\|^2_H\end{aligned} \quad \text{(A.8)}$$

where $\xi=\hat{x}^{(N)}-\bar{x}$. By (35) and (41)

$$\begin{aligned}d\hat{k}_i=&\Big\{-A^T\hat{k}_i-Q\hat{x}_i+(\Psi\bar{x}+\bar{\eta})+PR_2^{-1}(P\bar{x}+\bar{s})\\&-\bar{G}^Tv+KG(\hat{x}^{(N)}-\bar{x})-KR_2^{-1}(\hat{s}-\bar{s})\Big\}dt\\&+K\sigma dW_i,\ \hat{k}_i(T)=H\hat{x}_i(T).\end{aligned}$$

By (A.5) and Itô's formula

$$\sum_{i=1}^N \mathbb{E}[\hat{x}_i^T(T)H\tilde{x}_i(T)]$$
$$= \sum_{i=1}^N \mathbb{E}\int_0^T \Big\{-\big[Q\hat{x}_i - (\Psi\bar{x}+\bar{\eta}) - PR_2^{-1}(P\bar{x}+\bar{s})$$
$$+\bar{G}^T v - KG(\hat{x}^{(N)}-\bar{x}) + KR_2^{-1}(\hat{s}-\bar{s})\big]^T\tilde{x}_i$$
$$+\hat{k}_i^T(\bar{G}\tilde{x}^{(N)} + B\tilde{u}_i - R_2^{-1}\tilde{s})\Big\}dt$$

and

$$0 = \sum_{i=1}^N \mathbb{E}[l^T(T)\tilde{s}(T) - l^T(0)\tilde{s}(0)]$$
$$= N\mathbb{E}\int_0^T [R_2^{-1}v + R_2^{-1}(P\bar{x}+\bar{s})]^T\tilde{s}dt$$
$$- N\mathbb{E}\int_0^T (l^T PB\tilde{u}^{(N)})dt.$$

The above two equations lead to

$$\sum_{i=1}^N \mathbb{E}[\hat{x}_i^T(T)H\tilde{x}_i(T)]$$
$$= \sum_{i=1}^N \mathbb{E}\int_0^T \Big\{-\big[Q\hat{x}_i - (\Psi\bar{x}+\bar{\eta}) - PR_2^{-1}(P\bar{x}+\bar{s})$$
$$-KG(\hat{x}^{(N)}-\bar{x}) + KR_2^{-1}(\hat{s}-\bar{s})\big]^T\tilde{x}_i$$
$$-\hat{u}_i^T R_1\tilde{u}_i + (\hat{k}^{(N)}-v)^T\bar{G}\tilde{x}_i$$
$$-(\hat{k}^{(N)}-v)R_2^{-1}\tilde{s} + (P\bar{x}+\bar{s})^T R_2^{-1}\tilde{s}\Big\}dt.$$

From this and (A.8)

$$\sum_{i=1}^N I_i = \sum_{i=1}^N \mathbb{E}\int_0^T \big[\xi^T(\Psi - PR_2^{-1}P + KG)\tilde{x}_i$$
$$+ (\hat{k}^{(N)}-v)^T(\bar{G}\tilde{x}_i - R_2^{-1}\tilde{s})$$
$$+((P+K)\tilde{x}_i+\tilde{s})R_2^{-1}(\bar{s}-\hat{s})\big]dt.$$

By Lemmas 2.2, 2.5, and Schwarz inequality, we obtain

$$\frac{1}{N}\sum_{i=1}^N I_i = O\left(\frac{1}{\sqrt{N}}\right).$$

From this with (A.7), the theorem follows. ∎

Appendix B
Proof of Theorem 3.2

To prove Theorem 3.2, we need three lemmas.

Lemma 4.1: Assume that (A0), (A1), (A5)–(A8) hold. For Problem (PI), we have

$$\mathbb{E}\int_0^\infty e^{-\rho t}\left(\|\hat{x}^{(N)}-\bar{x}\|^2 + \|\hat{s}-\bar{s}\|^2\right)dt = O\left(\frac{1}{N}\right). \quad \text{(B.1)}$$

Proof. By a similar argument to (43)–(44), we obtain

$$d\xi = (\bar{A}+\bar{G})\xi dt - R_2^{-1}\chi dt + \frac{1}{N}\sum_{i=1}^N \sigma dW_i$$
$$\xi(0) = \frac{1}{N}\sum_{i=1}^N x_{i0} - \bar{x}_0$$
$$d\chi = -\left[(A+\bar{G})^T\chi + PBR_1^{-1}B^T K\xi\right]dt$$
$$+ \frac{1}{N}\sum_{i=1}^N\sum_{j=1}^N\left(\beta_i^j - \frac{\sigma}{N}\right)dW_j$$

where $\xi=\hat{x}^{(N)}-\bar{x}$ and $\chi=\hat{s}-\bar{s}$. By Itô's formula and (A8), we have $\chi = \tilde{P}\xi + \psi$, where $\tilde{P}$ is given by (65). Denote $\Upsilon = \bar{A}+\bar{G}-R_2^{-1}\tilde{P}$. Then

$$\xi(t) = e^{\Upsilon t}\xi(0) + \frac{1}{N}\int_0^t e^{\Upsilon(t-\mu)}\sum_{i=1}^N \sigma dW_i(\mu).$$

This with (A8) gives $\mathbb{E}\int_0^\infty e^{-\rho t}\|\xi(t)\|^2 dt = O(1/N)$. ∎

Lemma 4.2: Assume that (A0), (A1), (A5)–(A8) hold. For Problem (PI) and any N

$$\max_{1\le i\le N}\mathbb{E}\int_0^\infty e^{-\rho t}\left(\|\hat{x}_i\|^2 + \|\hat{u}_i\|^2\right)dt < \infty. \quad \text{(B.2)}$$

Proof: By (A7) and Lemma 6.1, we obtain that

$$\mathbb{E}\int_0^\infty e^{-\rho t}(\|\hat{x}^{(N)}(t)\|^2 + \|\hat{s}(t)\|^2)dt < \infty.$$

Note that $\bar{A}-\frac{\rho}{2}I$ is Hurwitz. By Schwarz's inequality

$$\mathbb{E}\int_0^\infty e^{-\rho t}\|\hat{x}_i(t)\|^2 dt$$
$$\le C + 3\mathbb{E}\int_0^\infty e^{-\rho\mu}\|g(\mu)\|^2\int_\mu^\infty t\left\|e^{(\bar{A}-\frac{\rho}{2}I)(t-\mu)}\right\|^2 dt d\mu$$
$$+ 3C\mathbb{E}\int_0^\infty e^{-\rho\mu}\|\sigma(\mu)\|^2\int_\mu^\infty \left\|e^{(\bar{A}-\frac{\rho}{2}I)(t-\mu)}\sigma\right\|^2 dt d\mu$$
$$\le C + 3C\mathbb{E}\int_0^\infty e^{-\rho\mu}\|g(\mu)\|^2 d\mu$$
$$+ 3C\mathbb{E}\int_0^\infty e^{-\rho\mu}\|\sigma(\mu)\|^2 d\mu \le C_1.$$

This with (A7) completes the proof. ∎

Lemma 6.3: Assume $A-\frac{\rho}{2}I$ is Hurwitz. Then

$$\mathbb{E}\int_0^\infty e^{-\rho t}\|\hat{k}^{(N)} - v\|^2 dt \le O(1/N).$$

Proof: By (60) and some elementary computations, we obtain $d\vartheta(t) = -(A-\frac{\rho}{2}I)^T\vartheta(t)dt$, where $\vartheta = v - K\bar{x} - \varphi$. This leads to $\vartheta(t) = e^{-(A-\frac{\rho}{2}I)t}\vartheta(0)$. Since $A-\frac{\rho}{2}I$ is Hurwitz, and $\vartheta \in C_{\rho/2}([0,\infty),\mathbb{R}^n)$, then we have $\vartheta(t)\equiv 0$, which implies $v = K\bar{x}+\varphi$. By Lemma 6.1, $\int_0^\infty e^{-\rho t}\mathbb{E}\|\hat{k}^{(N)}-v\|^2 dt \le O(1/N)$. This completes the proof. ∎

Proof of Theorem 3.2: As in the proof of Theorem 2.4, we restrict to Problem (P2′). It suffices to consider all $u_i \in \mathcal{U}_c$ such that $\sup_{f\in\mathcal{U}_c} J_{\rm soc}(u,f) \le \sup_{f\in\mathcal{U}_c} J_{\rm soc}(\hat{u},f) \le NC_0$. Taking

$f=0$, we have

$$\mathbb{E}\int_0^\infty e^{-\rho t}\|u_i\|^2 dt < C. \tag{B.3}$$

By (53) and [34], we have

$$\mathbb{E}\int_0^\infty e^{-\rho t}\|s\|^2 dt \le C_1 \mathbb{E}\int_0^\infty e^{-\rho t}\|u^{(N)}\|^2 dt < C.$$

Noticing $A+\bar{G}-\frac{\rho}{2}I$ is Hurwitz, one can obtain $\mathbb{E}\int_0^\infty e^{-\rho t}\|x^{(N)}\|^2 dt < C$ which with (B.3) implies

$$\mathbb{E}\int_0^\infty e^{-\rho t}\left(\|x_i\|^2+\|u_i\|^2+\|s\|^2\right) dt \le C. \tag{B.4}$$

From this and (B.2)

$$\mathbb{E}\int_0^\infty (\|\tilde{x}_i\|^2+\|\tilde{u}_i\|^2+\|\tilde{s}\|^2) dt < C. \tag{B.5}$$

We have $J_{\rm soc}(u)=\sum_{i=1}^N (J_i(\hat{u})+\tilde{J}_i(\tilde{u})+\mathcal{I}_i)$, where

$$\begin{aligned}\tilde{J}_i(\tilde{u}) &\triangleq \frac{1}{2}\mathbb{E}\int_0^\infty e^{-\rho t}\Big[\|\tilde{x}_i-\Gamma\tilde{x}^{(N)}\|_Q^2 \\ &\quad +\|\tilde{u}_i\|_{R_1}^2-\|P\tilde{x}^{(N)}\|_{R_2}^2\Big] dt\\ \mathcal{I}_i &\triangleq \mathbb{E}\int_0^\infty e^{-\rho t}\Big[\left(\hat{x}_i-\Gamma\hat{x}^{(N)}-\eta\right)^T Q\,(\tilde{x}_i \\ &\quad -\Gamma\tilde{x}^{(N)}\Big)+\hat{u}_i^T R_1\tilde{u}_i \\ &\quad -\left(P(\hat{x}^{(N)}+s)\right)^T R_2^{-1}(P\tilde{x}^{(N)}+\tilde{s})\Big] dt.\end{aligned}$$

From Lemma 3.1 and Proposition 3.1, $\tilde{J}_i(\tilde{u})\ge 0$ for $N\ge N_0$. By Itô's formula and straightforward computations

$$\begin{aligned}\sum_{i=1}^N \mathcal{I}_i &= \sum_{i=1}^N \mathbb{E}\int_0^\infty e^{-\rho t}\Big\{\tilde{x}_i^T\left[Q\hat{x}_i-\Psi\bar{x}-\bar{\eta}\right] \\ &\quad +\hat{u}_i^T R_1\tilde{u}_i-(P\bar{x}+\hat{s})^T R_2^{-1}P\tilde{x}_i\Big\} dt\\ &\quad +\sum_{i=1}^N \mathbb{E}\int_0^\infty e^{-\rho t}\xi^T[(\Psi-PR_2^{-1}P)\tilde{x}_i-PR_2^{-1}\tilde{s}]dt\\ &\quad -N\mathbb{E}\int_0^\infty e^{-\rho t}(P\bar{x}+\hat{s})^T R_2^{-1}\tilde{s}dt\\ &= \sum_{i=1}^N \mathbb{E}\int_0^\infty e^{-\rho t}\left[\xi^T(\Psi-PR_2^{-1}P+KG)\tilde{x}_i\right.\\ &\quad +(\hat{k}^{(N)}-v)^T(\bar{G}\tilde{x}_i-R_2^{-1}\tilde{s})\\ &\quad \left.+((P+K)\tilde{x}_i+\tilde{s})R_2^{-1}(\bar{s}-\hat{s})\right] dt.\end{aligned}$$

From (B.1) and (B.5), we obtain

$$\frac{1}{N}\sum_{i=1}^N \mathcal{I}_i = O(1/\sqrt{N}).$$

■

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